

LIE ALGEBRAS OF CONSERVATION LAWS OF VARIATIONAL PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We establish a version of the first Noether Theorem, according to which the (equivalence classes of) conserved quantities of given Euler-Lagrange equations in several independent variables are in one-to-one correspondence with the (equivalence classes of) vector fields satisfying an appropriate pair of geometric conditions, namely: (a) they preserve the class of vector fields tangent to holonomic submanifolds of a jet space; (b) they leave invariant the action, from which the Euler-Lagrange equations are derived, modulo terms identically vanishing along holonomic submanifolds. Such correspondence between symmetries and conservation laws is built on an explicit linear map Φ_α from the vector fields satisfying (a) and (b) into the conserved differential operators, and not into their divergences as it occurs in other proofs of Noether Theorem. This map Φ_α is not new: it is the map determined by contracting symmetries with a form of Poincaré-Cartan type α and it is essentially the same considered for instance in a paper by Kuper-shmidt. There it was shown that Φ_α determines a bijection between symmetries and conservation laws in a special form. Here we show that, if appropriate regularity assumptions are satisfied, *any* conservation law is equivalent to one that belongs to the image of Φ_α , proving that the corresponding induced map $\tilde{\Phi}_\alpha$ between equivalence classes of symmetries and equivalence classes of conservation laws is actually a bijection. All results are given coordinate-free formulations and rely just on basic differential geometric properties of finite-dimensional manifolds.

1. INTRODUCTION

In a previous paper ([6]) it was established a new version of the celebrated Noether Theorem on the bijection between (equivalence classes of) conservation laws and (equivalence classes of) symmetries of Euler-Lagrange equations for the case of functions of one independent variable.

The main purpose of that paper was to give a self-contained proof of Noether Theorem, in a coordinate free formulation and relying only on standard differential geometric properties of finite-dimensional manifolds. An

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outcome of this approach was the realization of the fact that Noether's correspondence between conservation laws and symmetries can be actually determined by a linear map that goes directly from the Lie algebra of infinitesimal symmetries into the vector space of constants of motions, and not into the space of their differentials as it occurs in other proofs of Noether Theorem. Such linear map is very simple: it is the map Φ_α that sends an infinitesimal symmetry X into the function $f := \iota_X \alpha$, where α is a fixed 1-form, determined by the Lagrangian that gives the Euler-Lagrange equations. This α is a generalisation of the Poincaré-Cartan 1-form $\alpha_H := p_i dq^i - H dt$ of Hamiltonian Mechanics.

We have to stress the fact that Φ_α is not new: for instance, it essentially coincides with the correspondence between symmetries and a special class of conservation laws, established by Kupershmidt in [9], Thm. II.5.1, for Lagrangians and Euler-Lagrange equations of arbitrary order and for field theories with an arbitrary number of independent variables. We emphasise that the main result in [6] asserts that *any* conserved quantity is, up to the addition of a trivially conserved quantity, equivalent to one contained in the image of the mapping Φ_α . This means that the induced correspondence $\tilde{\Phi}_\alpha$ between equivalence classes of symmetries and equivalence classes of conserved quantities is actually a bijection. Kupershmidt, in contrast, only shows that the conserved currents in a particular form, that is those obtained by a particular contraction of a vector field with a form of Poincaré-Cartan type, is contained in the image of the mapping Φ_α .

In this paper we extend the geometric construction of [6] to the general case of conservation laws and Euler-Lagrange equations for functions of m independent variables. All notions and arguments considered in [6] are directly extended to such general setting. Differences occur only in few points and are due only to the presence of a higher number of independent variables. Actually, during the preparation of this paper, we realised that in [6] the first and third author gave an incorrect claim, which is here removed. A detailed erratum for [6] is given in the appendix.

As in the previous paper, Noether's correspondence between symmetries and conservation laws is established by means of a linear map Φ_α , which transforms the elements X of the Lie algebra of infinitesimal symmetries into the conserved $(m-1)$ -forms $\eta = \iota_X \alpha$, where α is a fixed m -form, called *of Poincaré-Cartan type*. Here, with the expression "conserved $(m-1)$ -form" we mean an $(m-1)$ -form with components that constitute a vector valued differential operator with a divergence that vanishes on the solutions to the Euler-Lagrange equations. As mentioned above, this linear map Φ_α is essentially the same considered in [9], Ch. II.5 and, as in [6], our main Theorem 4.9 shows that, under appropriate regularity conditions, *any* conserved quantity is, up to the addition of a trivially conserved $(m-1)$ -form, equivalent to one contained in the image of the mapping Φ_α . Due to this, we get that the induced map $\tilde{\Phi}_\alpha$ between equivalence classes of symmetries

and equivalence classes of conserved quantities is a true bijection, improving in this way Kupershmidt's result in full generality.

In order to make as much as possible clear and explicit all aspects of innovation of our results, in §2 we overview Olver's version of Noether Theorem, which, at the best of our knowledge, is the most general and complete variant of this theorem (see [16, 17, 8, 18]). We then outline our results, pointing out differences and similarities with Olver's and other variants of Noether Theorem, as for instance those given in [13, 10, 3].

Structure of the paper. After section §2, where the reader can find an outline of all contents of this paper, in §3 we introduce the main ingredients of our approach, namely the notions of holonomic forms, variational classes and variational principles for actions defined by variational classes. In §4, we prove the first and second part of Noether Theorem: in the first, we show that, by contraction with a fixed m -form of Poincaré-Cartan type, any \mathcal{I} -symmetry is associated with a conserved $(m - 1)$ -form; in the second, we prove that, under appropriate regularity conditions, this correspondence can be reversed. In §5, an explicit example of an m -form of Poincaré-Cartan type is given. In Appendix, the above mentioned erratum for [6] is given.

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2. AN OUTLINE OF THE RESULTS AND COMPARISONS WITH PREVIOUS VERSIONS OF THE NOETHER THEOREM

2.1. A short overview of Olver's version of Noether Theorem.

Consider a system of partial differential equations of order k of class \mathcal{C}^∞

$$F_\nu \left(x^i, y^j, \frac{\partial y^j}{\partial x^\ell}, \dots, \frac{\partial^k y^j}{\partial x^{\ell_1} \dots \partial x^{\ell_k}} \right) = 0, \quad \nu = 1, \dots, N, \quad (2.1)$$

for n unknown functions $y^j(x^i)$ of m independent variables x^i , $1 \leq i \leq m$. An m -tuple of smooth differential operators of order r

$$P = (P^1, \dots, P^m), \quad P^\ell = P^\ell \left(x^i, y^j, \frac{\partial y^j}{\partial x^\ell}, \dots, \frac{\partial^r y^j}{\partial x^{\ell_1} \dots \partial x^{\ell_r}} \right)$$

is said to satisfy a conservation law for (2.1) if the equation

$$\text{Div} \left(P \left(x^i, y^j(x^s), \frac{\partial y^j}{\partial x^\ell} \Big|_{(x^s)}, \dots, \frac{\partial^r y^j}{\partial x^{\ell_1} \dots \partial x^{\ell_r}} \Big|_{(x^s)} \right) \right) = 0 \quad (2.2)$$

is identically satisfied whenever $y^j(x^s)$ is a solution to (2.1). If the m -tuple P is identically vanishing on all solutions of (2.1), the conservation law is called *trivial of the first kind*. If (2.2) holds for *all* smooth maps $y^j(x^s)$ (not just for the solutions to (2.1)), the conservation law is called *trivial of the second kind*. We shortly call *trivial conservation law* any sum of such two types of conservation laws.

Given a non-negative integer s , the *prolongation of (2.1) to order $k + s$* is the p.d.e.'s system that is determined by the equations in (2.1) together with their derivatives up to order s . It is therefore a system

$$F_\nu^{(s)}\left(x^i, y^j, \frac{\partial y^j}{\partial x^\ell}, \dots, \dots, \frac{\partial^r y^j}{\partial x^{\ell_1} \dots \partial x^{\ell_r}}, \dots, \frac{\partial^{k+s} y^j}{\partial x^{\ell_1} \dots \partial x^{\ell_{k+s}}}\right) = 0, \quad (2.3)$$

where now ν runs from 1 up to an appropriate integer $N_{k+s} \geq N_k := N$, which depends on the order s of the prolongation.

Now, it is possible to show that if the map $F^{(r+1-k)} = (F_\nu^{(r+1-k)})_{1 \leq \nu \leq N_{r+1}}$ locally satisfies an appropriate constant rank condition, then for any $(r+1)$ -th order differential operator of the form $\text{Div}(P)$, which appear in a conservation law (2.2), there locally exists a set of differential operators of order $(r+1)$

$$Q^\nu = Q^\nu\left(x^i, y^j, \frac{\partial y^j}{\partial x^\ell}, \dots, \frac{\partial^r y^j}{\partial x^{\ell_1} \dots \partial x^{\ell_r}}\right), \quad 1 \leq \nu \leq N_{r+1},$$

such that (see [16], formula (4.27))

$$\text{Div}(P) = \sum_{\rho=1}^{N_{r+1-k}} Q^\rho F_\rho^{(r+1-k)}. \quad (2.4)$$

The operators Q^ν are determined by the operator $\text{Div}(P)$ up to addition of a differential operator that vanishes identically on all solutions to (2.1).

Assume now that (2.1) is a system of Euler-Lagrange equations, that is a system of equations that characterises the stationary points, within the class of local variations with fixed boundary values, of a functional

$$\mathcal{I} = \int_{\mathcal{U}} L\left(x^i, y^j, \frac{\partial y^j}{\partial x^\ell}, \dots, \frac{\partial^{k'} y^j}{\partial x^{\ell_1} \dots \partial x^{\ell_{k'}}}\right) dx^1 \wedge \dots \wedge dx^m \quad (2.5)$$

for some smooth L , usually called *Lagrangian* (or *Lagrangian density*). Note that a Lagrangian L can be also considered as a smooth real valued function on the infinite jet space $J^\infty(\mathbb{R}^m; \mathbb{R}^n)$.

We now recall that there is a special class of vector fields on $J^\infty(\mathbb{R}^m; \mathbb{R}^n)$, called *variational symmetries of \mathcal{L}* , whose associated 1-parameter groups of (local) diffeomorphisms satisfy the following conditions (see [16], Ch. 5):

- a) they leave invariant the set of maps

$$j^\infty(\sigma) : \mathcal{U} \subset \mathbb{R}^m \longrightarrow J^\infty(\mathbb{R}^m; \mathbb{R}^n),$$

given by the jets $j_x^\infty(\sigma)$ of \mathcal{C}^∞ local maps $\sigma : \mathcal{U} \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$;

- b) they transform L into other Lagrangians L' that differs from L by terms that give trivial contributions to the Euler-Lagrange equations.

A vector field of this kind is called *trivial variational symmetry* if it vanishes on the jets $j_x^\infty(\sigma)$ of the solutions $\sigma : \mathcal{U} \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$ to the Euler-Lagrange equations (2.1). Two variational symmetries are said to be *equivalent* if

they differ by a trivial one. It is known that any equivalence class contains a subclass of elements \mathbf{v}_Q in special form, each of them uniquely determined by a special n -tuple of differential operators $Q = (Q^1, \dots, Q^n)$. Any such element is called *variational symmetry in evolutionary form*.

Olver's proof of Noether Theorem is crucially based on the following

Theorem 2.1. *Let L be a Lagrangian of order k' and $F_\nu = 0$, $1 \leq \nu \leq n$, its associated system of Euler-Lagrange equations of order $k = 2k' + 1$. Suppose also that for any r its prolonged p.d.e. system $F^{(r-k)} = (F_\nu^{(r-k)}) = 0$ satisfies appropriate constant rank conditions.*

Then a given (locally defined) m -tuple of smooth differential operators $P = (P^1, \dots, P^m)$ of order r satisfies a conservation law for the Euler-Lagrange equations $F_\nu = 0$ if and only if it is equivalent (i.e. it differs by an m -tuple satisfying a trivial conservation law) to an m -tuple \tilde{P} , whose divergence $\text{Div } \tilde{P}$ has the form

$$\text{Div } \tilde{P} = \sum_{\rho=1}^n Q^\rho F_\rho \quad (2.6)$$

where $Q = (Q^1, \dots, Q^n)$ is the n -tuple associated with a variational symmetry \mathbf{v}_Q of L in evolutionary form.

We remark that the constant rank condition on $F^{(r-k)}$ is needed just in the proof of the “only if” part and that, for *any* Euler-Lagrange equation, a variational symmetry always determine a conservation law.

From this result the following general version of Noether Theorem follows.

Noether Theorem. *If L is a Lagrangian having prolongations of the associated Euler-Lagrange equations satisfying appropriate conditions on ranks, local solvability and existence of non-characteristic directions (more precisely, they are normal and totally nondegenerate systems; see [16] for definitions), then there exists a one-to-one correspondence between*

- a) *conservation laws for the Euler-Lagrange equations of L , determined up to additions of trivial conservation laws;*
- b) *variational symmetries of L , determined up to additions of trivial variational symmetries.*

This version of Noether Theorem is based on the map between symmetries and conservation laws determined by (2.6). Note that such map goes from the space of variational symmetries in evolutionary form to the space of divergences, not into the space of the conserved m -tuples $P = (P^i)$.

2.2. An outline of our approach.

2.2.1. Holonomic submanifolds and holonomic distributions on jet spaces.

Consider a bundle $\pi : E \longrightarrow M$ over an m -dimensional oriented manifold M . Since all our discussions are of purely local nature, for simplicity, from

now on we assume that $M = \mathbb{R}^m$, oriented by the standard volume form $\omega = dx^1 \wedge \dots \wedge dx^m$.

For a given k -th order jet space $\pi^k : J^k(E) \rightarrow \mathbb{R}^m$, any (local) section $\sigma : \mathcal{U} \subset \mathbb{R}^m \rightarrow E$ is uniquely associated with the submanifold $(j^k \sigma)(\mathcal{U})$ of $J^k(E)$, given by their k -th order jets $j_x^k(\sigma)$, $x \in \mathcal{U}$. These submanifolds are usually called *holonomic* ([5]) and can be characterised as the only m -dimensional submanifolds of $J^k(E)$ with:

- a) maximal rank projections onto M ;
- b) all tangent spaces are contained in the vector spaces of a special distribution $\mathcal{D} \subset TJ^k(E)$.

Being related with the holonomic sections, we call such \mathcal{D} the *holonomic distribution of $J^k(E)$* . Note that in other places such distribution is called differently, as for instance *canonical differential system* ([21, 22]) or *Cartan distribution* ([7, 3]).

A (locally defined) r -form λ on $J^k(E)$ is called *holonomic* if

- a) either $0 \leq r \leq m$ and λ vanishes when it is evaluated on r vector fields in \mathcal{D} or
- b) $r \geq m + 1$ and λ vanishes when it is evaluated on at least m vector fields in \mathcal{D} .

If we set $s := \min\{r, m\}$, we may also say that an r -form λ is holonomic if and only if its restriction to an m -dimensional holonomic submanifold vanishes identically when it is evaluated on at least s vector fields that are tangent to such submanifold.

2.2.2. Lagrangians and actions.

We now observe that any functional on the class of sections of $\pi : E \rightarrow \mathbb{R}^m$ of the form $\mathcal{I} = \int_{\mathcal{U}} L(j_x^k(\sigma)) dx^1 \wedge \dots \wedge dx^m$ can be considered as a functional on the class of (oriented) holonomic submanifolds of $J^k(E)$, defined by

$$\mathcal{I}|_{j^k \sigma(\mathcal{U})} := \int_{j^k(\sigma)(\mathcal{U})} \alpha_L, \quad \alpha_L := L dx^1 \wedge \dots \wedge dx^m. \quad (2.7)$$

Here we use the notation $\int_S \alpha_L$ to indicate the integral of the restriction of α_L to the tangent space of S .

The following fact is a crucial ingredient of our construction (see §3.3): in the class of fixed boundary variations a holonomic submanifold $j^k(\sigma)(\mathcal{U})$ is a stationary point for \mathcal{I} if and only if it is a stationary point for any other functional $\mathcal{I}' = \int_{j^k(\sigma)(\mathcal{U})} (\alpha_L + \lambda + d\mu)$ with λ and μ holonomic. Due to this, we say that two m -forms α, α' on $J^k(E)$ are *variationally equivalent* if $\alpha - \alpha' = \lambda + d\mu$ for some λ and μ holonomic and we observe that a variational principle for (2.7) can be considered as uniquely associated with the variational equivalence class $[\alpha_L]$ of α_L .

2.2.3. Conserved quantities as differential forms.

Consider an m -tuple of smooth r -th order differential operators $P = (P^1, \dots, P^m)$ and the associated $(m-1)$ -form on $J^r(E)$

$$\eta_P = \sum_{j=1}^m (-1)^{m-1} P^j dx^1 \wedge \dots \widehat{}_j \dots \wedge dx^m. \quad (2.8)$$

Given a section $\sigma : \mathcal{U} \rightarrow E$, one can check that $\text{Div } P|_{j^k \sigma(\mathcal{U})} = 0$ if and only if the restriction $d\eta_P|_{T(j^k \sigma(\mathcal{U}))}$ of the differential $d\eta_P$ to the tangent space of $j^r \sigma(\mathcal{U}) \subset J^r(E)$ is identically equal to 0. Further, one has (see §4.1):

- (1) $d\eta_P|_{T(j^r \sigma(\mathcal{U}))} = 0$ if and only if $d\eta'|_{T(j^r \sigma(\mathcal{U}))} = 0$ for any $(m-1)$ -form $\eta' = \eta_P + \mu + d\nu$ with μ, ν holonomic;
- (2) the integrals of η_P and $\eta' = \eta_P + \mu + d\nu$ on any closed $(m-1)$ -dimensional submanifold of a holonomic submanifold are equal.

This motivates the following definitions. We say that two $(m-1)$ -forms η, η' on $J^r(E)$ are *variationally equivalent* if $\eta - \eta' = \mu + d\nu$ for some holonomic forms μ, ν . Moreover, given an m -tuple of r -th order differential operators $P = (P^i)$, we call *variational class of P* the equivalence class $[\eta_P]$ of $(m-1)$ -forms on $J^r(E)$ that are variationally equivalent to η_P .

By (1) and (2), P satisfies a conservation law for a differential system if and only if the differential of an $(m-1)$ -form η in the variational class $[\eta_P]$ vanishes identically when restricted to the tangent spaces of the holonomic submanifolds associated with solutions.

2.2.4. Infinitesimal \mathcal{I} -symmetries and Noether Theorem.

Let L be a smooth Lagrangian on $J^k(E)$ and \mathcal{I} the functional (2.7) on holonomic submanifolds. We call *weak (infinitesimal) symmetry for \mathcal{I}* or, shortly, *weak \mathcal{I} -symmetry* any vector field on $J^k(E)$ that generates a 1-parameter group of (local) diffeomorphisms which

- (1) preserve the holonomic distribution \mathcal{D} or, more precisely, a slightly weaker condition, namely they map a special subset of the vector fields in \mathcal{D} into vector fields in \mathcal{D} (see details in Definition 4.2), and
- (2) map an element $\alpha \in [\alpha_L]$ into m -forms of the same variational class.

Using coordinates, one can check that the vector fields on $J^\infty(\mathbb{R}^m, \mathbb{R}^n)$ satisfying (1) and (2) coincide with the vector fields that Olver calls *variational symmetries*. Hence our weak \mathcal{I} -symmetries can be considered as finite-dimensional versions (defined in a coordinate free language) of Olver's variational symmetries. We also have to mention that even the vector fields that are called *Noether symmetries* in [3] are related with our weak \mathcal{I} -symmetries. In fact, using coordinates, one can check that they locally coincide with Olver's *variational symmetries in evolutionary form*. Hence, they correspond to a special subclass of our weak \mathcal{I} -symmetries.

Our main result is the following (Theorems 4.7 and 4.9).

Theorem 2.2. *Let $L : J^k(E) \rightarrow \mathbb{R}$ be a smooth Lagrangian that depends only on jets components of order k' satisfying the inequality $2k' + 2 \leq [\frac{k}{2}]$. Then there exists an m -form α in the variational class of $\alpha_L = Ldx^1 \wedge \dots \wedge dx^m$ with the following properties.*

- i) *For any (weak) \mathcal{I} -symmetry X on $J^k(E)$, the $(m-1)$ -form $\eta = \iota_X \alpha$ is associated with an m -tuple $P = (P^i)$ of k -th order differential operators satisfying a conservation law for the Euler-Lagrange equations of L .*
- ii) *Let $k_o \leq [\frac{k}{2}] - 1$ and $\mathcal{W} \subset J^k(E)$ be an open subset of the domain of α where the Euler-Lagrange equations $\mathcal{E}(L) = 0$ of L have a prolonged system with appropriate conditions on ranks and on the family of jets of its solutions. For any m -tuple of k_o -th order differential operators $P = (P^i)$ on \mathcal{W} , satisfying a conservation law for $\mathcal{E}(L) = 0$, there exists a weak \mathcal{I} -symmetry X such that*

$$\iota_X \alpha = \eta_P + \mathfrak{z}_{P'}$$

where η_P is defined in (2.8) and $\mathfrak{z}_{P'}$ is an $(m-1)$ -form corresponding to an m -tuple $P' = (P'^i)$ satisfying a trivial conservation law.

As we mentioned in the Introduction, the m -form α is called of *Poincaré-Cartan type* (see 4.5 for details) and it corresponds to the form $S\Omega$ defined by Kupershmidt in [9], §II.3. The proof of Prop. A2 in [19] (see also [4], Thm.1.3.11) provides an algorithm to determine an m -form of Poincaré-Cartan type for any given Lagrangian.

2.3. Comparisons with previous versions of Noether Theorem.

The above Theorem 2.2 yields the existence of a one-to-one correspondence between equivalence classes of weak \mathcal{I} -symmetries and equivalence classes of conservation laws, exactly as it is implied by Olver's Theorem 2.1 or other versions of Noether Theorem (see e.g. [3], §5.4.1). On the other hand, in our approach such correspondence is determined by means of a very simple linear map, namely the contraction map $X \mapsto \iota_X \alpha$ with an m -form α of Poincaré-Cartan type. This gives a direct way to go from the weak \mathcal{I} -symmetries of k -th order into conserved m -tuples P of k_o -th order operators, *not into the space of divergence operators* as it occurs in Olver's and other versions of Noether Theorem. Further, this map is surjective, in the sense that *any* conserved m -tuple P is, modulo addition of m -tuples satisfying trivial conservation laws, in the image of the above described linear map.

Another result of our approach is the unveiling of the importance of a distinguished relation between the Poincaré-Cartan 1-form of Hamiltonian Mechanics and conservation laws, a relation that generalises to *all* smooth systems of ordinary and partial differential equations of variational origin. We also point out that all notions considered in our construction are expressed in terms of standard differential geometric objects. The proofs use

only basic properties of differential forms on finite-dimensional manifolds, as for instance Stokes' Theorem and Homotopy Formula. This paves the way to direct extensions of Noether Theorem to many other interesting settings, as e.g. to supergeometric contexts. We plan to undertake this task in future papers.

We conclude recalling that a direct correspondence between symmetries and conserved quantities was also established by Lychagin for the Euler-Lagrange equations that are in the class of *Monge-Ampère equations*. This is a large and important family of non-linear second order differential equations on real functions $f : \mathcal{U} \subset \mathbb{R}^m \rightarrow \mathbb{R}$ of m independent variables (see [13, 10] and references therein). They are equations usually denoted by $\Delta_\omega(f) = 0$ and they are equivalent to the vanishing of some fixed k -form ω on $J^1(E)$, $E = \mathbb{R} \times \mathbb{R}^m$, on the holonomic submanifold $j^1(f)(\mathcal{U})$ of the unknown function $f : \mathcal{U} \subset \mathbb{R}^m \rightarrow \mathbb{R}$. In the cases in which $\Delta_\omega(f) = 0$ coincides with an Euler-Lagrange equation, Lychagin constructed an explicit linear map from the class of symmetries of the equation into the class of conserved quantities, which establishes the bijection of Noether Theorem ([13], Thm. 4.4). We expect that Lychagin's map coincides with our map $X \mapsto \iota_X \alpha$ for an appropriate choice of an m -form α of Poincaré-Cartan type.

We observe that Lychagin's map can be constructed for *all Monge-Ampère equations of divergence type*, not only for those of variational origin. We expect that a deeper understanding of the relation between m -forms of Poincaré-Cartan type and Lychagin's map would lead to interesting generalisations of Noether Theorem.

3. A DIFFERENTIAL-GEOMETRIC PRESENTATION OF VARIATIONAL PRINCIPLES

3.1. Notational remarks.

In what follows, we consider only partial differential equations on \mathcal{C}^∞ maps from open subsets of \mathbb{R}^m , oriented by the standard volume form $dx^1 \wedge \dots \wedge dx^m$, into a fixed n -dimensional manifold M . Since any such map $f : \mathcal{U} \subset \mathbb{R}^m \rightarrow M$ is uniquely determined by the associated (local) section of the trivial bundle $\pi : E = \mathbb{R}^m \times M \rightarrow \mathbb{R}^m$

$$\sigma^{(f)}(x^1, \dots, x^m) := (x^1, \dots, x^m, f(x^1, \dots, x^m)) ,$$

we always consider a system of partial differential equations as a set of differential equations on the smooth sections of the bundle E .

Given an integer $k \geq 1$ and a smooth section $\sigma : \mathcal{U} \subset \mathbb{R}^m \rightarrow E = \mathbb{R}^m \times M$, we denote by $j_p^k(\sigma)$ for the k -th order jet of σ at $p \in \mathcal{U}$. The space of all k -jets is denoted by $J^k(E)$. For any $1 \leq \ell \leq k$, we set

$$\pi_\ell^k : J^k(E) \rightarrow J^\ell(E) , \quad \pi_\ell^k(j_p^k(\sigma)) := j_p^\ell(\sigma)$$

and we denote by $\pi_0^k : J^k(E) \longrightarrow E$ and $\pi_{-1}^k : J^k(E) \longrightarrow \mathbb{R}^m$ the natural projections onto E and \mathbb{R}^m , i.e. the maps

$$\pi_0^k(j_p^k(\sigma)) := \sigma(p) \quad \text{and} \quad \pi_{-1}^k(j_p^k(\sigma)) := p, \quad \text{respectively}.$$

Given a section $\sigma : \mathcal{U} \subset \mathbb{R}^m \longrightarrow E$, we call *k-th order lift* of σ the map

$$\sigma^{(k)} : \mathcal{U} \subset \mathbb{R}^m \longrightarrow J^k(E), \quad \sigma^{(k)}(p) := j_p^k(\sigma).$$

Finally, given a system of coordinates $\xi = (y^i) : \mathcal{W} \subset M \longrightarrow \mathbb{R}^n$ on an open set $\mathcal{W} \subset M$, we denote by $\widehat{\xi}$ the associated coordinates

$$\widehat{\xi} : \mathbb{R}^m \times \mathcal{W} \longrightarrow \mathbb{R}^{n+m}, \quad \widehat{\xi}(p, q) := (x^1(p), \dots, x^m(p), y^1(q), \dots, y^n(q)),$$

where the x^i 's are the standard coordinates of \mathbb{R}^m . The coordinates $\widehat{\xi} = (x^i, y^j)$ are called *associated with the coordinates* $\xi = (y^i)$. Any set of coordinates constructed in this fashion is called *set of adapted coordinates*.

For a given set of adapted coordinates (x^i, y^j) , we may consider the naturally associated set of coordinates

$$\widehat{\xi}^{(k)} = \left(x^i, y^j, (y_I^j)_{|I|=1, \dots, k} \right) : \mathcal{U} \subset J^k(E) \longrightarrow \mathbb{R}^{m+n+N},$$

$$N := n \sum_{\ell=1}^k \binom{m+\ell-1}{\ell}, \quad (3.1)$$

defined for any $u = j_p^k(\sigma)$ in $\mathcal{U} = (\pi_0^k)^{-1}(\mathbb{R}^m \times \mathcal{W})$ as follows:

- a) the coordinates $x^i(u)$, $1 \leq i \leq m$, are the standard coordinates of $p = \pi_{-1}^k(u) \in \mathbb{R}^m$;
- b) the coordinates $y^j(u)$, $1 \leq j \leq n$, are the last n coordinates of the set of adapted coordinates of $(p, s(p)) = \pi_0^k(u) \in \mathbb{R}^m \times M$;
- c) the coordinates $y_I^j(u)$, with $1 \leq j \leq m$ and $I = (I_1, \dots, I_m)$ multi-index of order $|I| := \sum_{j=1}^m I_j$ with $1 \leq |I| \leq k$, are the values of the partial derivatives

$$y_I^j(u) := \left. \frac{\partial^{|I|} \sigma^j}{\partial x^I} \right|_{(x^1(p), \dots, x^m(p))}$$

of a section σ in the equivalence class $u = j_p^k(\sigma)$.

The coordinates $\widehat{\xi}^{(k)}$ are called *adapted coordinates on $J^k(E)$ associated with the coordinates $\xi = (y^i)$* .

3.2. Holonomic p-forms and variational classes.

Definition 3.1. The *holonomic submanifolds of $J^k(E)$* (see e.g. [5]) are the submanifolds $\mathcal{S} \subset J^k(E)$ for which there exists a section $\sigma : \mathcal{U} \subset \mathbb{R}^m \longrightarrow E$ such that $\mathcal{S} = \{ u \in J^k(E) : u = \sigma^{(k)}(x), x \in \mathcal{U} \}$.

We call *holonomic distribution of $J^k(E)$* the distribution $\mathcal{D} \subset TJ^k(E)$ generated at any $u \in J^k(E)$ by the vectors that are tangent to holonomic submanifolds, i.e.,

$$\mathcal{D}_u = \text{Span} \left\{ v \in T_u J^k E : v = \sigma_*^{(k)}(w) \text{ for some } w \in T_p \mathbb{R}^m \right. \\ \left. \text{and some } \sigma \text{ such that } j_p^k(\sigma) = u \right\}.$$

The vectors in \mathcal{D} and the vector fields with values in \mathcal{D} are called *holonomic* (see §2 for other names often used for the distribution \mathcal{D}).

Let $\widehat{\xi}^{(k)} = (x^i, y^j, y_I^j)$ be a set of adapted coordinates on some open set $\mathcal{U} \subset J^k(E)$ and fix a jet $\bar{u} = j_p^k(\bar{\sigma})$ with coordinates $\widehat{\xi}^{(k)}(\bar{u}) = (\bar{x}^i, \bar{y}^j, \bar{y}_I^j)$. The vectors $v \in T_{\bar{u}} J^k(E)$ having the form $v = \sigma_*^{(k)}(w)$ for some $w = w^i \frac{\partial}{\partial x^i} \Big|_p \in T_p \mathbb{R}^m$ and some section σ with $j_p^k(\sigma) = \bar{u}$, are

$$v = w^i \left(\frac{\partial}{\partial x^i} \Big|_{(\bar{x}^i, \bar{y}^j, \bar{y}_I^j)} + \sum_{0 \leq |I| \leq k-1} \bar{y}_{I+1_i}^j \frac{\partial}{\partial y_I^j} \Big|_{(\bar{x}^i, \bar{y}^j, \bar{y}_I^j)} + \right. \\ \left. + \sum_{|J|=k} \frac{\partial^{|J|+1} \sigma^j}{\partial x^{J+1_i}} \Big|_{\bar{x}} \frac{\partial}{\partial y_J^j} \Big|_{(\bar{x}^i, \bar{y}^j, \bar{y}_I^j)} \right). \quad (3.2)$$

(here, given $J = (J_1, \dots, J_m)$, we set $J + 1_i := (J_1, \dots, J_i + 1, \dots, J_m)$).

Since the values $\frac{\partial^{|J|+1} \sigma^j}{\partial x^{J+1_i}} \Big|_{\bar{x}}$, $|J| = k$, may vary arbitrarily by making different choices for σ in the k -th order jet $\bar{u} = j_p^k(\sigma)$, we have that $\mathcal{D}_u \subset T_u J^k(E)$ is generated by the linearly independent vectors

$$\frac{d}{dx^i} \Big|_{(\bar{x}^i, \bar{y}^j, \bar{y}_I^j)} := \left(\frac{\partial}{\partial x^i} + \sum_{0 \leq |I| \leq k-1} \bar{y}_{I+1_i}^j \frac{\partial}{\partial y_I^j} \right) \Big|_{(\bar{x}^i, \bar{y}^j, \bar{y}_I^j)} \\ \text{and} \quad \frac{\partial}{\partial y_J^j} \Big|_{(\bar{x}^i, \bar{y}^j, \bar{y}_I^j)} \quad \text{with } |J| = k. \quad (3.3)$$

The notion of holonomic distribution leads to the following.

Definition 3.2. A (local) p -form λ of $J^k(E)$ is called *holonomic* if it satisfies one of the following conditions:

- a) $p \leq m$ and for any p -tuple (X_1, \dots, X_p) of holonomic vector fields, one has $\lambda(X_1, \dots, X_p) = 0$;
- b) $p > m$ and for any m -tuple (X_1, \dots, X_m) of holonomic vector fields, one has $\lambda(X_1, \dots, X_m, \star, \dots, \star) = 0$.

If α, α' are p -forms on the same open subset $\mathcal{U} \subset J^k(E)$, we call them *variationally equivalent* if there exist a holonomic p -form λ and a holonomic

$(p-1)$ -form μ such that

$$\alpha' = \alpha + \lambda + d\mu .$$

For a fixed $\mathcal{U} \subset J^k(E)$, the variational equivalence is an equivalence relation on the set of p -forms on \mathcal{U} . The equivalence class of α is called *variational class of α* and is denoted by $[\alpha]$.

Finally, we say that a p -form α is *proper* if $\iota_V \alpha = 0$ for any vector field V that is vertical with respect to the projection $\pi_{k-1}^k : TJ^k(E) \rightarrow TJ^{k-1}(E)$.

The role played by holonomic forms and variational classes in our approach has been shortly described in §2. See §3.3 below for further details.

The explicit expressions in coordinates of holonomic q -forms is quite helpful to get a better understanding of these objects. To write them down, we first need to impose the following order on the set of indices:

- a) the multiindices are subjected to the lexicographic order, namely given $J = (J_1, \dots, J_m)$ and $J' = (J'_1, \dots, J'_m)$, we say that $J < J'$ if and only if $|J| < |J'|$ or $|J| = |J'|$ and there exists $\ell \leq m$ such that $j_i = j'_i$ for $i = 1, \dots, \ell-1$ and $j_\ell < j'_\ell$;
- b) given two pairs (j, J) and (j', J') with $1 \leq j, j' \leq n$ and J, J' multiindices, we write $(j, J) < (j', J')$ to indicate that either $J < J'$ or $J = J'$ and $j < j'$.

Consider now the collection of 1-forms

$$\begin{aligned} dx^i & \quad \text{for } 1 \leq i \leq m , \\ \omega_J^j &:= dy_J^j - \sum_{\ell=1}^m y_{J+1_\ell}^j dx^\ell \quad \text{for } 1 \leq j \leq n , \ 0 \leq |J| \leq k-1 , \\ \psi_L^j &:= dy_L^j \quad \text{for } 1 \leq j \leq n , \ |L| = k . \end{aligned} \quad (3.4)$$

This collection of 1-forms gives a basis for $T_u^* J^k(E)$ at any u , so that any q -form α can be written as a linear combination of wedge products of such 1-forms and it can be written as

$$\sum_{\substack{\ell+r+s=q \\ i_1 < \dots < i_\ell \\ (j_1, J_1) < \dots < (j_r, J_r) \\ (k_1, L_1) < \dots < (k_s, L_s) \\ 0 \leq |J_s| \leq k-1 , \ |L_j| = k}} \alpha_{i_1 \dots i_\ell | j_1 \dots j_r | k_1 \dots k_s}^{J_1 \dots J_r | L_1 \dots L_s} dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge \omega_{J_1}^{j_1} \wedge \dots \wedge \omega_{J_r}^{j_r} \wedge \psi_{L_1}^{k_1} \wedge \dots \wedge \psi_{L_s}^{k_s} . \quad (3.5)$$

From this expression and the definition of $\psi_{J'}^j$, we see that α is proper if and only if it is of the form

$$\alpha = \sum_{\substack{\ell+r=q \\ i_1 < \dots < i_\ell \\ (j_1, J_1) < \dots < (j_r, J_r) \\ 0 \leq |J_s| \leq k-1 ,}} \alpha_{i_1 \dots i_\ell | j_1 \dots j_r}^{J_1 \dots J_r} dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge \omega_{J_1}^{j_1} \wedge \dots \wedge \omega_{J_r}^{j_r} . \quad (3.6)$$

On the other hand, α is holonomic if and only if it is determined by an expression (3.5) satisfying one of these conditions:

- a') $q < m$ and all coefficients of terms with $\ell + s = q$ (hence, with $r = 0$) are equal to 0;
- b') $q \geq m$ and all coefficients of terms with $\ell + s \geq m$ (hence, with $r \leq q - m$) are equal to 0.

Consequently a *proper* q -form α is holonomic if and only if it admits one of the following expressions:

a'') $q < m$ and

$$\alpha = \sum_{\substack{\ell+r=q \\ 1 \leq r \leq q}} \sum_{\substack{i_1 < \dots < i_\ell \\ (j_1, J_1) < \dots < (j_r, J_r) \\ 0 \leq |J_s| \leq k-1}} \alpha_{i_1 \dots i_\ell | j_1 \dots j_r}^{J_1 \dots J_r} dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge \omega_{J_1}^{j_1} \wedge \dots \wedge \omega_{J_r}^{j_r} ;$$

b'') $q \geq m$ and

$$\alpha = \sum_{\substack{\ell+r=q \\ q-m+1 \leq r \leq q}} \sum_{\substack{i_1 < \dots < i_\ell \\ (j_1, J_1) < \dots < (j_r, J_r) \\ 0 \leq |J_s| \leq k-1}} \alpha_{i_1 \dots i_\ell | j_1 \dots j_r}^{J_1 \dots J_r} dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge \omega_{J_1}^{j_1} \wedge \dots \wedge \omega_{J_r}^{j_r} .$$

These formulae are quite useful to quickly check several properties of holonomic forms. For instance, one can directly see that *the differential $d\alpha$ of a holonomic form α needs not be holonomic*.

3.3. Variational classes, Lagrangians and source forms.

We now consider variational principles for functionals of the form

$$I_L(\sigma) = \int_{\mathcal{U}} (L \circ \sigma^{(k)})(x^1, \dots, x^m) dx^1 \wedge \dots \wedge dx^m , \quad (3.7)$$

determined by a smooth Lagrangian $L : J^k(E) \rightarrow \mathbb{R}$. As it was explained in [19] (see also [4, 6]), the functionals (3.7) can be considered as special cases of a slightly larger class of functionals, which we now recall.

Definition 3.3. Let $[\alpha]$ be a variational class of m -forms in $J^k(E)$. We call *action determined by $[\alpha]$* the functional on sections $\sigma : \mathcal{U} \subset \mathbb{R}^m \rightarrow E$ on regions \mathcal{U} with piecewise smooth boundary $\partial\mathcal{U}$, defined by

$$\mathcal{I}_{[\alpha]}(\sigma) := \int_{\sigma^{(k)}(\mathcal{U})} \alpha . \quad (3.8)$$

Here, we use the notation $\int_{\sigma^{(k)}(\mathcal{U})} \alpha$ to indicate the integral of an m -form α in the variational class $[\alpha]$, restricted to the points and to the tangent vectors of the oriented m -dimensional submanifold $\sigma^{(k)}(\mathcal{U})$ of $J^k(E)$.

We stress the fact that, by the very definition of variational classes, *the integral $\int_{\sigma^{(k)}(\mathcal{U})} \alpha$ is independent on the choice of the representative α in $[\alpha]$* and it is therefore well defined. Indeed, if α, α' are variational equivalent,

i.e. $\alpha' = \alpha + \lambda + d\mu$ for some holonomic m -form λ and holonomic $(m-1)$ -form μ , by Stokes' Theorem and the fact that the vectors that are tangent to $\sigma^{(k)}(\mathcal{U})$ are holonomic,

$$\int_{\sigma^{(k)}(\mathcal{U})} \alpha' = \int_{\sigma^{(k)}(\mathcal{U})} \alpha + \int_{\sigma^{(k)}(\mathcal{U})} \lambda + \int_{\partial\sigma^{(k)}(\mathcal{U})} \mu = \int_{\sigma^{(k)}(\mathcal{U})} \alpha .$$

Furthermore, *any functional of the form (3.7) can be considered as an action of the form (3.8)*. Indeed, if $L : J^k(E) \rightarrow \mathbb{R}$ is a function of class \mathcal{C}^∞ and if we set $\alpha_L := L \pi_{-1}^*(dx^1 \wedge \dots \wedge dx^m)$, we see that for any section σ

$$\mathcal{I}_{[\alpha_L]}(\sigma) = \int_{\sigma^{(k)}(\mathcal{U})} \alpha_L = \int_{\mathcal{U}} (L \circ \sigma^{(k)}) dx^1 \wedge \dots \wedge dx^m = I_L(\sigma) .$$

Conversely, *any action having the form (3.8) can be locally identified with a functional of the form (3.7)*. To see this, let α be an m -form on $J^k(E)$ and consider the pull-back $\tilde{\alpha} := (\pi_k^{k+1})^*(\alpha)$ on $J^{k+1}(E)$. Being $\tilde{\alpha}$ proper, its expression in adapted coordinates $\tilde{\xi}^{(k)} = (x^i, y^j, y_I^j)$ has the form (see (3.6)):

$$\begin{aligned} \tilde{\alpha} &= \alpha_0 dx^1 \wedge \dots \wedge dx^m + \\ &+ \sum_{\ell=1}^{m-1} \left(\sum_{\substack{i_1 < \dots < i_\ell \\ (j_1, J_1) < \dots < (j_{m-\ell}, J_{m-\ell}) \\ 0 \leq |J_s| \leq k-1,}} \alpha_{i_1 \dots i_\ell | j_1 \dots j_{m-\ell}}^{J_1 \dots J_{m-\ell}} dx^{i_1} \wedge \dots \wedge dx^{i_\ell} \wedge \omega_{J_1}^{j_1} \wedge \dots \wedge \omega_{J_{m-\ell}}^{j_{m-\ell}} \right). \end{aligned} \quad (3.9)$$

We now observe that here all terms except the first one are holonomic. Hence $[\tilde{\alpha}] = [\alpha_0 dx^1 \wedge \dots \wedge dx^m]$. This means that the values of $\mathcal{I}_{[\alpha]}$ on sections with values in the domain \mathcal{W} of the adapted coordinates coincide with those given by the functional (3.7) with $L := \alpha_0|_{\mathcal{W}}$.

These remarks show that *the class of functionals introduced with Definition 3.3 is a natural extension of the class of usual actions (3.7)*.

We conclude introducing the following convenient terminology. Let $\tilde{\beta}$ be a (locally defined) p -form on a jet space $J^k(E)$. We say that $\tilde{\beta}$ is of order r if we can write $\tilde{\beta} = (\pi_r^k)^* \beta$ for some (locally defined) p -form β on $J^r(E)$, $0 \leq r \leq k$. According to this definition, *any p -form β on a jet space $J^r(E)$ can be naturally identified with a p -form of order r on any other jet space $J^k(E)$ with $k \geq r+1$* . Further, note that *if a p -form β' on $J^k(E)$ is of order $0 \leq r \leq k-1$, then it is proper*.

Due to this it is possible to identify any (not necessarily proper) p -form on a jet space $J^r(E)$ with a *proper* p -form (of order r) on a jet space $J^k(E)$ with $k \geq r+1$. This shows that, in many arguments, there is no loss of generality if one reduces to consider only proper q -forms.

3.4. Variational Principles and Euler-Lagrange equations.

We now consider variational principles for the actions defined in Definition 3.3. As the reader will shortly see, our presentation is designed to derive from a given variational principle the same Euler-Lagrange equations that one obtains from Lagrangians in usual settings.

Consider a section $\sigma : \mathcal{U} \rightarrow E$ and a regular m -dimensional region D in $\mathcal{U} \subset \mathbb{R}^m$. With the expression “regular region” we mean a connected open subset $D \subset \mathcal{U}$, whose closure \overline{D} is an m -dimensional oriented manifold with corners (see e.g. [11] for the definition).

A smooth map $F : D \times (-\varepsilon, \varepsilon) \rightarrow E$ is called *variation of $\sigma|_D$ with fixed k -th order boundary* if it satisfies the following conditions:

- a) the maps $F^{(s)} := F(\cdot, s) : D \rightarrow E$, $s \in (-\varepsilon, \varepsilon)$, are such that $F^{(0)} = \sigma$ and, for any s , the map $F^{(s)}$ is smoothly extendible to \overline{D} ;
- b) for any $s \in (-\varepsilon, \varepsilon)$, the k -order lift $(F^{(s)})^{(k)} := j^k(F^{(s)})$ of the extension $F^{(s)} : \overline{D} \rightarrow E$ satisfies the boundary condition $(F^{(s)})^{(k)}|_{\partial D} = \sigma^{(k)}|_{\partial D}$.

Definition 3.4. Let $[\alpha]$ be a variational class of m -forms on $J^k(E)$ and $\sigma : \mathcal{U} \subset \mathbb{R}^m \rightarrow E$ a section. We say that σ *satisfies the variational principle of $\mathcal{I}_{[\alpha]}$* if for any regular region $D \subset \mathcal{U}$ and any variation F of $\sigma|_D$ with fixed k -th order boundary, one has

$$\left. \frac{d(\mathcal{I}_{[\alpha]}(F^{(s)}))}{ds} \right|_{s=0} = \frac{d}{ds} \left(\int_{j^k(F^{(s)})(D)} \alpha \right) \Big|_{s=0} = 0 . \quad (3.10)$$

We now want to show that the sections that satisfy such variational principle are precisely the solutions to the usual Euler-Lagrange equations of classical setting. For this, we first need to reformulate (3.10) into an equivalent condition involving a special kind of vector fields.

Let $\sigma : \mathcal{U} \rightarrow E$ be a section, $D \subset \mathcal{U}$ a regular region and $W : \sigma^{(k)}(\overline{D}) \rightarrow TJ^k(E)|_{\sigma^{(k)}(\overline{D})}$ a vector field defined only at the points of $\sigma^{(k)}(\overline{D})$. We say that W is a *k -th order variational field* if there exists a smooth variation $F : D \times (-\varepsilon, \varepsilon) \rightarrow E$ of σ with fixed k -th order boundary such that

$$W = F_*^{(k)} \left(\frac{\partial}{\partial s} \Big|_{(x,0)} \right) , \quad F^{(k)}(x, s) := j_x^k(F(\cdot, s)) . \quad (3.11)$$

We remark that, by the property (b) of the variations with fixed k -th order boundary, any variational vector field W is such that

$$W|_{\sigma^{(k)}(\partial D)} = 0 . \quad (3.12)$$

Proposition 3.5. *A section $\sigma : \mathcal{U} \rightarrow E$ satisfies the variational principle of $\mathcal{I}_{[\alpha]}$ if and only if*

$$\int_{\sigma^{(k)}(D)} \iota_W d\alpha = 0 \quad (3.13)$$

for any regular region $D \subset \mathcal{U}$ and any k -th order variational field W on $\sigma^{(k)}(\overline{D})$.

Proof. Let W be the variational field (3.11) determined by a smooth variation F with fixed k -th order boundary. By Stokes' Theorem for manifolds with corners (see e.g. [12]) and by (3.12), for any m -form α on $J^k(E)$ and any $h \in (-\varepsilon, \varepsilon)$

$$\begin{aligned} \int_{(F(h))^{(k)}(D)} \alpha - \int_{(F(0))^{(k)}(D)} \alpha &= \int_{D \times \{h\}} F^{(k)*}(\alpha) - \int_{D \times \{0\}} F^{(k)*}(\alpha) = \\ &= \int_{\partial(D \times (0, h))} F^{(k)*}(\alpha) = \int_{D \times (0, h)} F^{(k)*}(d\alpha) . \end{aligned}$$

Hence,

$$\begin{aligned} \left. \frac{d(\mathcal{I}_{[\alpha]}(F(s)))}{ds} \right|_{s=0} &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\int_{(F(h))^{(k)}(D)} \alpha - \int_{(F(0))^{(k)}(D)} \alpha \right) = \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \int_{D \times (0, h)} F^{(k)*}(d\alpha) = \int_D \sigma^{(k)*}((-1)^m \iota_W d\alpha) = \\ &= (-1)^m \int_{\sigma^{(k)}(D)} \iota_W d\alpha . \end{aligned}$$

From this the claim follows. \square

In absence of an effective characterisation of the variational vector fields, condition (3.13) does not seem to correspond to any system of partial differential equations for σ . On the other hand, we have to stress that if σ satisfies (3.13) for a given choice of variational vector field W , it also satisfies the equality $\int_{\sigma^{(k)}(D)} \iota_W \beta = 0$ for any $(m+1)$ -form β which is variationally equivalent to $d\alpha$. Indeed, if $\beta = d\alpha + \lambda + d\mu$ for some holonomic λ and μ , by Stokes' Theorem, holonomicity and (3.12), we have

$$\begin{aligned} \int_{\sigma^{(k)}(D)} \iota_W d\alpha &= \int_{\sigma^{(k)}(D)} \iota_W \beta - \int_{\sigma^{(k)}(D)} \iota_W d\mu = \\ &= \int_{\sigma^{(k)}(D)} \iota_W \beta - \int_{\sigma^{(k)}(D)} \mathcal{L}_W \mu + \int_{\sigma^{(k)}(D)} d(\iota_W \mu) = \\ &= \int_{\sigma^{(k)}(D)} \iota_W \beta - \int_{\sigma^{(k)}(D)} \mathcal{L}_W \mu + \int_{\sigma^{(k)}(\partial D)} \iota_W \mu = \\ &= \int_{\sigma^{(k)}(D)} \iota_W \beta - \int_{\sigma^{(k)}(D)} \mathcal{L}_W \mu . \end{aligned} \tag{3.14}$$

Here, $\mathcal{L}_W \mu$ is to be understood as the Lie derivative of μ along some smooth extension of W on a neighbourhood of $\sigma^{(k)}(\overline{D})$. By definition of W , we may always assume that such local extension has a local flow Φ_t^W , which is the lift to $J^k(E)$ of a fiber preserving flow $\Phi_t^{\widetilde{W}}$ on E , generated by a local vector field \widetilde{W} of E that projects trivially on \mathbb{R}^m . Under this assumption, the local flow Φ_t^W maps holonomic sections into holonomic sections, hence it preserves the

holonomic distribution \mathcal{D} . This yields that the Lie derivatives of holonomic forms by W are holonomic and that $\int_{\sigma^{(k)}(D)} \iota_W d\alpha = \int_{\sigma^{(k)}(D)} \iota_W \beta$, as claimed.

This fact motivates the importance of some special representatives of $[d\alpha]$, called *source forms* and which are now about to define. For this we need to introduce a preliminary notion (see e.g. [19, 4]): A proper q -form β on a jet space $J^k(E)$, $k \geq 1$, is called *homogeneous* if there are non-negative integers ℓ, r such that $\ell + r = q$ and so that, for any set $\{X_1, \dots, X_q\}$ of q vector fields that contains either more than ℓ holonomic vector fields or more than r vector fields projecting trivially on \mathbb{R}^m , one has

$$\beta(X_1, \dots, X_q) = 0.$$

If β is homogeneous and satisfies the above condition for the integers ℓ and r , we call the pair (ℓ, r) the *bi-degree* of β . It can be checked that the bi-degree of a non-trivial proper homogeneous q -form β is uniquely associated with β .

Definition 3.6. A *source form* on $J^k(E)$ is any (locally defined) $(m+1)$ -form β which is proper, homogeneous of bi-degree $(m, 1)$ and such that

$$\beta(X_1, \dots, X_m, V) = 0 \quad (3.15)$$

for any holonomic vector fields X_i and any π_0^k -vertical vector field V (i.e., such that $\pi_{0*}^k(V) = 0$).

For a better understanding of source forms, it is convenient to see what are the coordinate expressions of these $(m+1)$ -forms in a system of adapted coordinates $\widehat{\xi}^{(k)} = (x^i, y^j, y_I^j)$. One can directly check that an $(m+1)$ -form β is a source form if and only if it has the form

$$\beta = \sum_{j=1}^n \beta_j dx^1 \wedge \dots \wedge dx^m \wedge dy^j = \sum_{j=1}^n \beta_j dx^1 \wedge \dots \wedge dx^m \wedge \omega_0^j \quad (3.16)$$

at all points where the coordinates are defined. We also remark that by Prop. A.2 in [19] (see also [20, 4, 6]) *given an m -form $\alpha = L dx^1 \wedge \dots \wedge dx^m$ determined by a Lagrangian L of order r , the variational class $[d\alpha]$ on a jet space $J^k(E)$ with $k \geq 2r$ contains exactly one source form β . Locally, such source form is given by the coordinate expression (3.16) in which the components β_j are determined by applying the classical Euler-Lagrange operator to L . In particular, when $r = 1$ and $k = 2$, the explicit expressions of the components β_j are*

$$\beta_j := -\frac{\partial L}{\partial y^j} + \sum_{\ell=1}^m \frac{d}{dx^\ell} \left(\frac{\partial L}{\partial y_\ell^j} \right) \quad (3.17)$$

(for properties of higher order Lagrangians and Euler-Lagrange operators, see e.g. [2], §II.B).

We are now able to show that a section satisfies a variational principle for $\mathcal{I}_{[\alpha]}$ if and only if it satisfies the corresponding Euler-Lagrange equations.

Theorem 3.7. *Let $\alpha = Ldx^1 \wedge \dots \wedge dx^m$ be an m -form of order r on a jet space $J^k(E)$ with $k \geq 2r$ and β the unique source form in $[d\alpha]$. Then $\sigma : \mathcal{U} \rightarrow E$ satisfies the variational principle of $\mathcal{I}_{[\alpha]}$ if and only if, for any $u \in \sigma^{(k)}(\mathcal{U})$ and $v \in T_u J^k(E)|_{\mathcal{U}}$,*

$$\iota_v \beta|_u(X_1, \dots, X_m) = 0 \quad \text{for any choice of vectors } X_i \in T_u(\sigma^{(k)}(\mathcal{U})). \quad (3.18)$$

Proof. By Proposition 3.5 and the remark after (3.14), a section $\sigma : \mathcal{U} \rightarrow E$ satisfies the variational principle if and only if

$$\int_{\sigma^{(k)}(D)} \iota_W \beta = 0 \quad (3.19)$$

for any regular domain $D \subset \mathcal{U}$ and any k -th order variational field W . If D is sufficiently small, so that $\sigma^{(k)}(D)$ is included in the domain of a set of adapted coordinates $\xi^{(k)} = (x^i, y^j, y_I^j)$, we may write

$$W = W^j \frac{\partial}{\partial y^j} + \sum_{|I|=1}^k W_I^j \frac{\partial}{\partial y_I^j} \quad \text{and} \quad \iota_W \beta = (W^i \beta_i) dx^1 \wedge \dots \wedge dx^m. \quad (3.20)$$

We also observe that, for any given choice of maps $f^i : \sigma^{(k)}(D) \rightarrow \mathbb{R}$, $i = 1, \dots, n$, that vanish identically on a neighbourhood of ∂D , one can construct a smooth variation F with fixed boundary up to order k , whose associated variational field W has coordinate components given by

$$W^i|_{\sigma^{(k)}(p)} = f^i|_{\sigma^{(k)}(p)} \quad \text{for any } p \in D.$$

This and (3.20) yield that (3.19) is satisfied for any regular domain D and any choice of W if and only if the restrictions $\beta_i|_{\sigma^{(k)}(\mathcal{U})}$ are identically vanishing. This means that σ is a solution if and only if the m -form

$$\iota_W \beta = (-1)^m W^i \beta_i dx^1 \wedge \dots \wedge dx^m$$

is identically vanishing on $\sigma^{(k)}(\mathcal{U})$ for *any* choice of a vector field $W = W^j \frac{\partial}{\partial y^j} + \sum_{|I|=1}^k W_I^j \frac{\partial}{\partial y_I^j}$ at the points of $\sigma^{(k)}(\mathcal{U})$ (and *not just for vector fields W of variational type*). The claim follows. \square

From this proof and the remarks before (3.17), we directly see that whenever the action $\mathcal{I}_{[\alpha]}$ is determined by a Lagrangian density L , condition (3.18) holds if and only if the section σ satisfies the usual system of Euler-Lagrange equations determined by L , as claimed.

4. A NEW PROOF OF THE NOETHER THEOREM

4.1. Conservation laws for a system of variational p.d.e.'s.

We call *p-form-valued differential operator of order k* a smooth map $\eta : J^k(E) \rightarrow \Lambda^p T^* \mathbb{R}^m$ which makes the following diagram commute (here, $\pi : \Lambda^p T^* \mathbb{R}^m \rightarrow \mathbb{R}^m$ is the standard projection)

$$\begin{array}{ccc}
J^k(E) & \xrightarrow{\eta} & \Lambda^p T^* \mathbb{R}^m \\
\pi_{k,-1} \searrow & & \swarrow \pi \\
& \mathbb{R}^m &
\end{array}$$

Any such map has necessarily the form (2.8) for some appropriate m -tuple $P = (P^j)$ of smooth maps $P^j : J^k(E) \rightarrow \mathbb{R}$. We call P the *m -tuple of differential operators associated with η* . Note also that, for any section $\sigma : \mathcal{U} \rightarrow E$ defined on an open set $\mathcal{U} \subset \mathbb{R}^m$, the map $\eta|_{\sigma^{(k)}} := \eta \circ \sigma^{(k)} : \mathcal{U} \rightarrow \Lambda^p T^* \mathbb{R}^m$ is a smooth p -form on $\mathcal{U} \subset \mathbb{R}^m$.

Consider an $(m-1)$ -valued differential operator $\eta : J^k(E) \rightarrow \Lambda^{m-1} T^* \mathbb{R}^n$ and a variational class $[\alpha]$ of m -forms on $J^k(E)$. We say that η satisfies a *conservation law for $\mathcal{I}_{[\alpha]}$* if for any section $\sigma : \mathcal{U} \rightarrow E$ that satisfies the variational principle of $\mathcal{I}_{[\alpha]}$, one has

$$\int_{\partial D} \eta|_{\sigma^{(k)}} = 0 \quad (4.1)$$

on all boundaries ∂D of regular regions D in the domain \mathcal{U} of σ . One can directly check that this holds if and only if the associated m -tuple $P = (P^j)$ satisfies (2.2) for all solutions of the variational principle.

We now want to express condition (4.1) in terms of variational classes. Let $\tilde{\eta}$ be the $(m-1)$ -form on $J^k(E)$ defined by

$$\tilde{\eta}|_u := (\pi_{-1}^k)^* \eta(u) \quad \text{for any } u \in J^k(E). \quad (4.2)$$

By construction, for any section $\sigma : \mathcal{U} \rightarrow E$ and any regular domain $D \subset \mathcal{U}$,

$$\int_{\partial D} \eta|_{\sigma^{(k)}} = \int_{\sigma^{(k)}(\partial D)} \tilde{\eta}.$$

Further, for any $(m-1)$ -form $\tilde{\eta}'$, which is in the same variational class of $\tilde{\eta}$ (i.e. $\tilde{\eta}' = \tilde{\eta} + \lambda + d\mu$ for some λ, μ holonomic), we have

$$\int_{\sigma^{(k)}(\partial D)} \tilde{\eta}' = \int_{\sigma^{(k)}(\partial D)} \tilde{\eta} + \int_{\sigma^{(k)}(\partial D)} \lambda + \int_{\sigma^{(k)}(D)} d^2 \mu \stackrel{\lambda \text{ is holon.}}{=} \int_{\sigma^{(k)}(\partial D)} \tilde{\eta}.$$

This shows that (4.1) can be actually identified with an integral that depends only of the variational class of (4.2). Conversely, given an arbitrary $(m-1)$ -form $\tilde{\eta}'$ on $J^k(E)$ and an open set $\mathcal{U} \subset \mathbb{R}^m$ for which one can determine adapted coordinates on $\mathcal{W} := J^k(E|_{\mathcal{U}})$, one can directly determine an $(m-1)$ -form-valued differential operator $\eta : J^k(E|_{\mathcal{U}}) \rightarrow \Lambda^{m-1} T^* \mathbb{R}^n$ such that the $(m-1)$ -form (4.2) and the restriction $\tilde{\eta}'|_{(\pi_{-1}^k)^{-1}(\mathcal{U})}$ are in the same variational class. These observations motivate the following

Definition 4.1. Let α and η be an m -form and an $(m-1)$ -form, respectively, on $J^k(E)$. We say that the variational class $[\eta]$ satisfies a *conservation law*

for the action $\mathcal{I}_{[\alpha]}$ if for any section $\sigma : \mathcal{U} \longrightarrow E$ that satisfies the variational principle of $\mathcal{I}_{[\alpha]}$, one has $\int_{\sigma^{(k)}(\partial D)} \eta = 0$ for any regular domain $D \subset \mathcal{U}$.

By previous remarks, the conservation laws satisfied by $(m-1)$ -form-valued differential operators determine conservation laws satisfied by variational classes of $(m-1)$ -forms. At a local level, the converse is also true.

4.2. \mathcal{I} -Symmetries.

As announced in §2.2.4, our version of Noether Theorem is based on the following notions of “symmetry”.

Definition 4.2. Let X and α be a vector field and an m -form, respectively, defined on an open subset \mathcal{U} of $J^k(E)$.

- a) X is an *infinitesimal symmetry of the holonomic distribution* \mathcal{D} (shortly, *\mathcal{D} -symmetry*) if for all holonomic vector field Y on \mathcal{U} , the Lie derivative $\mathcal{L}_X Y$ is a holonomic vector field.
- b) X is a *weak \mathcal{D} -symmetry* if, for any holonomic vector field Y on \mathcal{U} and any $u \in \mathcal{U}$, there exists a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ of u and a holonomic vector field Y' on \mathcal{U}' such that 1) $\pi_{k-1}^k(Y') = 0$ and 2) the Lie derivative $\mathcal{L}_X(Y - Y')$ is holonomic.
- c) X is an *infinitesimal (weak) symmetry for $\mathcal{I}_{[\alpha]}$* (shortly, *(weak) \mathcal{I} -symmetry*) if it is a (weak) \mathcal{D} -symmetry and $\mathcal{L}_X \alpha$ is holonomic for some proper $\alpha \in [\alpha]$.

The notion of \mathcal{D} -symmetry is the direct generalisation of the corresponding definition considered in [6]. There, the discussion was limited to the case of jet spaces of maps of one independent variable, but most of their properties remain true in our more general situation. We briefly recall the main properties of \mathcal{D} -symmetries and refer to [6] for other details.

- 1) If a vector field X on an open subset $\mathcal{U} \subset J^k(E)$ is a \mathcal{D} -symmetry, then its local flow is a 1-parameter family of local diffeomorphisms mapping any holonomic submanifold $\sigma^{(k)}(\mathcal{U})$ into another submanifold, which is also locally holonomic, i.e. of the form $\sigma'^{(k)}(\mathcal{U}')$ for some other section $\sigma' : \mathcal{U}' \rightarrow E$.
- 2) If X is a \mathcal{D} -symmetry and λ is a holonomic p -form, then the local flow Φ_t^X of X is such that all local p -forms $\Phi_t^{X*}(\lambda)$, $t \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$, are holonomic. Hence, also the Lie derivative $\mathcal{L}_X \lambda$ is holonomic.
- 3) If α, α' are in the same variational class (i.e. $\alpha - \alpha' = \lambda + d\mu$, with λ, μ holonomic), then $\mathcal{L}_X \alpha$ is holonomic if and only if $\mathcal{L}_X \alpha'$ is holonomic.

The class of weak \mathcal{D} -symmetries is new and it naturally includes all \mathcal{D} -symmetries. This weaker version of the \mathcal{D} -symmetries is needed to remove an incorrect claim of [6] (see Appendix). We remark that if one works on the infinite order jet space $J^\infty(E)$ instead of the finite order jet space $J^k(E)$, the notions of \mathcal{D} -symmetry and weak \mathcal{D} -symmetry coincide.

4.3. Differential equations that characterise \mathcal{D} -symmetries and infinitesimal symmetries of an action.

Proposition 4.3. *Let X and α be a vector field and an m -form, respectively, on an open subset $\mathcal{U}' \subset J^k(E)$ and assume that $\widehat{\xi}^{(k)} = (x^i, y^j, y_J^j)$ are adapted coordinates on $\mathcal{U} \subset \mathcal{U}'$. Then:*

- (1) *A necessary condition for $X|_{\mathcal{U}}$ to be a \mathcal{D} -symmetry is that it satisfies the differential equations*

$$\omega_I^i \left(\mathcal{L}_X \frac{d}{dx^j} \right) = 0 \quad (4.3)$$

for all $1 \leq j \leq m$, $1 \leq i \leq n$ and $0 \leq |I| \leq k-1$. Conversely, if X satisfies the above system of differential equations, then it is a weak \mathcal{D} -symmetry.

- (2) *A necessary condition for $X|_{\mathcal{U}}$ to be an infinitesimal symmetry for $\mathcal{I}_{[\alpha]}$ (considered as functional on the sections of $E|_{\pi_0^k(\mathcal{U})}$) is that for some $\alpha_o \in [\alpha|_{\mathcal{U}}]$ it satisfies the system of differential equations (4.3) together with the differential equations*

$$(\mathcal{L}_X \alpha_o) \left(\frac{d}{dx^{i_1}}, \dots, \frac{d}{dx^{i_{m-r}}}, \frac{\partial}{\partial y_{J_1}^{j_1}}, \dots, \frac{\partial}{\partial y_{J_r}^{j_r}} \right) = 0 \quad (4.4)$$

for all $0 \leq r \leq m-1$, $1 \leq i_h \leq m$, $1 \leq j_\ell \leq n$ and $|J_\ell| = k$. Conversely, if X satisfies the systems (4.3) and (4.4) for some $\alpha_o \in [\alpha|_{\mathcal{U}}]$, then $X|_{\mathcal{U}}$ is a infinitesimal weak symmetry for $\mathcal{I}_{[\alpha]}$.

Proof. (1) Recall that $\mathcal{D}|_{\mathcal{U}}$ is generated by the vector fields $\frac{d}{dx^i}$, $1 \leq i \leq m$, and the vector fields $\frac{\partial}{\partial y_J^j}$, $1 \leq j \leq n$, $|J| = k$. It therefore consists of the intersections of the kernels of the 1-forms ω_I^i , $0 \leq |I| \leq k-1$, $1 \leq i \leq n$, at all points $u \in \mathcal{U}$. Hence $X|_{\mathcal{D}}$ is a \mathcal{D} -symmetry only if (4.3) holds. Conversely, assume that X satisfies the system of differential equations (4.3) and let Y be a holonomic vector field, i.e.

$$Y = Y^j \frac{d}{dx^j} + \sum_{1 \leq j \leq n, |J|=k} Y_J^j \frac{\partial}{\partial y_J^j}.$$

Then, $Y' := \sum_{1 \leq j \leq n, |J|=k} Y_J^j \frac{\partial}{\partial y_J^j}$ is a holonomic vector field such that: a) $\pi_{k-1*}^k(Y') = 0$; b) $\omega_I^i(\mathcal{L}_X(Y - Y')) = 0$ for any $1 \leq i \leq m$ and $1 \leq |I| \leq k-1$. This means that $\mathcal{L}_X(Y - Y')$ is holonomic and proves that X is a weak \mathcal{D} -symmetry.

(2) From (1) and definition of infinitesimal symmetry for $\mathcal{I}_{[\alpha]}$, it follows that the systems (4.3) and (4.4) are necessary conditions for X to be a \mathcal{D} -symmetry. The converse claim is a consequence of definitions and (1). \square

4.4. Explicit expressions for weak \mathcal{D} -symmetries.

Let $\widehat{\xi}^{(k)} = (x^j, y^i, y_J^i)$ be a system of adapted coordinates on an open subset $\mathcal{U} \subset J^k(E)$. For a given smooth map

$$\mathbf{v} = (\mathbf{v}_B^1, \dots, \mathbf{v}_B^m, \mathbf{v}^1, \dots, \mathbf{v}^n) : \mathcal{U} \subset J^k(E) \longrightarrow \mathbb{R}^{m+n},$$

let us denote by $X_{\mathbf{v}}$ the vector field

$$X_{\mathbf{v}} := \mathbf{v}_B^j \frac{\partial}{\partial x^j} + \mathbf{v}^i \frac{\partial}{\partial y^i} + \sum_{1 \leq i \leq n, 1 \leq |J| \leq k} \mathbf{v}_J^i \frac{\partial}{\partial y_J^i}, \quad (4.5)$$

where we follow the standard Einstein convention on summation over repeated indices and, for any multiindex $J = (J_1, \dots, J_m)$ with $1 \leq |J| \leq k$, we set

$$\mathbf{v}_J^i := \left(\frac{d}{dx^1} \right)^{J_1} \cdots \left(\frac{d}{dx^m} \right)^{J_m} (\mathbf{v}^i - y_{1r}^i \mathbf{v}_B^r) + y_{J+1r}^i \mathbf{v}_B^r. \quad (4.6)$$

(in this formula, we assume $y_J^i := 0$ when $|J| = k+1$). This yields that

$$\begin{aligned} X_{\mathbf{v}} &= \mathbf{v}_B^j \frac{d}{dx^j} + (\mathbf{v}^i - y_{1r}^i \mathbf{v}_B^r) \frac{\partial}{\partial y^i} + \\ &\quad + \sum_{1 \leq |J| \leq k} \left(\frac{d}{dx^1} \right)^{J_1} \cdots \left(\frac{d}{dx^m} \right)^{J_m} (\mathbf{v}^i - y_{1r}^i \mathbf{v}_B^r) \frac{\partial}{\partial y_J^i}. \end{aligned} \quad (4.7)$$

We may now prove the following

Proposition 4.4. *Given a set of adapted coordinates $\widehat{\xi}^{(k)} = (x^j, y^i, y_J^i)$ on an open subset $\mathcal{U} \subset J^k(E)$, the vector fields $X_{\mathbf{v}}$ defined in (4.5) are exactly the vector fields that satisfy the system of differential equations (4.3). In particular, all of them are weak \mathcal{D} -symmetries.*

Proof. Let $X = X_B^j \frac{\partial}{\partial x^j} + X^i \frac{\partial}{\partial y^i} + \sum_{1 \leq i \leq n, 1 \leq |J| \leq k} X_J^i \frac{\partial}{\partial y_J^i}$. Since

$$\mathcal{L}_X \frac{d}{dx^j} = -\frac{dX_B^r}{dx^j} \frac{\partial}{\partial x^r} + \sum_{0 \leq |J| \leq k-1} \left(X_{J+1j}^i - \frac{dX_J^i}{dx^j} \right) \frac{\partial}{\partial y_J^i} - \sum_{|J|=k} \frac{dX_J^i}{dx^j} \frac{\partial}{\partial y_J^i},$$

the system (4.3) is equivalent to

$$0 = \omega_I^i \left(\mathcal{L}_X \frac{d}{dx^j} \right) = X_{I+1j}^i - \frac{dX_I^i}{dx^j} + y_{I+1r}^i \frac{dX_B^r}{dx^j} \quad (4.8)$$

for all $1 \leq i \leq n$ and $0 \leq |I| \leq k-1$. This means that if X is a solution to (4.3), then the components X_J^i with $1 \leq |J| \leq k$ are uniquely determined by an inductive process from the components X_B^r and X^j . If we set $\mathbf{v}_B^r := X_B^r$ and $\mathbf{v}^j := X^j$, a straightforward check shows that $X = X_{\mathbf{v}}$. The last claim follows from Proposition 4.3. \square

4.5. The Noether Theorem.

Definition 4.5. We say that an m -form α on an open set $\mathcal{U} \subset J^k(E)$ is of *Poincaré-Cartan type* if its differential $d\alpha$ is equal to a source form up to addition of an holonomic $(m+1)$ -form.

The main motivation for this terminology comes from the fact that the well-known Poincaré-Cartan form $\alpha = p_i dq^i - H dt$ of Hamiltonian Mechanics is a 1-form of Poincaré-Cartan type according to the above definition (see [6] for details; see also [2], Ch. 5B, for other generalisations of the Poincaré-Cartan 1-form). We also remark that if α_L is an m -form on $\mathcal{U} \subset J^k(E)$ with adapted coordinate expression $\alpha_L = L dx^1 \wedge \dots \wedge dx^m$ for some Lagrangian L of order $r \leq \lfloor \frac{k}{2} \rfloor$, then for any $u \in \mathcal{U}$ there exists a neighbourhood $\mathcal{U}' \subset \mathcal{U}$ of u such that the variational class $[\alpha_L|_{\mathcal{U}'}]$ contains at last one 1-form of Poincaré-Cartan type. To see this, one needs only to consider a system of adapted coordinates on a neighbourhood \mathcal{U}' of u and the source form $\beta \in [d\alpha|_{\mathcal{U}'}]$ in (3.16), which has components determined by the Euler-Lagrange operator applied to L . Then $\beta = d\alpha_L|_{\mathcal{U}'} + d\mu + \lambda = d(\alpha_L|_{\mathcal{U}'} + \mu) + \lambda$ for some holonomic μ and λ and $\alpha := \alpha_L|_{\mathcal{U}'} + \mu$ is the required m -form of Poincaré-Cartan type in $[\alpha_L|_{\mathcal{U}'}]$.

We finally observe that the previous argument shows that if L is a Lagrangian of order r and $k > 2r$, the variational class $[\alpha_L|_{\mathcal{U}'}]$ contains an m -form which is not only of Poincaré-Cartan type, but also of order $k_o \leq k-1$. This additional condition is quite useful and it will be often required in the following.

The notion of m -forms of Poincaré-Cartan type leads to the following characterisation of weak \mathcal{I} -symmetries. As in Proposition 4.4, we consider a fixed adapted coordinates $\widehat{\xi}^{(k)} = (x^j, y^i, y_{(a)}^i)$ on an open set $\mathcal{U} \subset J^k(E)$.

Proposition 4.6. *Let α be an m -form of Poincaré-Cartan type of order $k_o \leq k-1$ and X a weak \mathcal{D} -symmetry on $\mathcal{U} \subset J^k(E)$ satisfying the system (4.3). Then X is such that $\mathcal{L}_X \alpha$ is holonomic (thus, an infinitesimal weak symmetry for $\mathcal{I}_{[\alpha]}$) if and only if it satisfies the following system of linear differential equation for some source form $\beta \in [d\alpha|_{\mathcal{U}}]$:*

$$\begin{aligned} \mathcal{L}_X \alpha \left(\frac{d}{dx^{i_1}}, \dots, \frac{d}{dx^{i_r}}, \frac{\partial}{\partial y_{j_1}^{j_1}}, \dots, \frac{\partial}{\partial y_{j_{m-r}}^{j_{m-r}}} \right) = \\ = d(\iota_X \alpha) \left(\frac{d}{dx^{i_1}}, \dots, \frac{d}{dx^{i_r}}, \frac{\partial}{\partial y_{j_1}^{j_1}}, \dots, \frac{\partial}{\partial y_{j_{m-r}}^{j_{m-r}}} \right) = 0, \end{aligned} \quad (4.9)$$

$$d(\iota_X \alpha) \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) = -\beta \left(X, \frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) \quad (4.10)$$

for all $1 \leq r \leq m-1$ and all choices of indices and multiindices with $1 \leq i_h \leq m$, $1 \leq j_\ell \leq r$ and $|J_\ell| = k$.

Proof. Let λ be the holonomic $(m+1)$ -form $\lambda := d\alpha - \beta$. From definitions, for any $0 \leq r \leq m$, $1 \leq i_\ell \leq m$, $1 \leq j_h \leq n$, $|J_h| = k$, we have

$$\lambda \left(\frac{d}{dx^{i_1}}, \dots, \frac{d}{dx^{i_r}}, \frac{\partial}{\partial y_{J_1}^{j_1}}, \dots, \frac{\partial}{\partial y_{J_{m-r}}^{j_{m-r}}}, \star \right) = 0, \quad \iota_{\frac{\partial}{\partial y_{J_1}^{j_1}}} \beta = 0.$$

From this and Proposition 4.3 (2), it follows that the weak \mathcal{D} -symmetry X is an infinitesimal symmetry for $\mathcal{I}_{[\alpha]}$ if and only if (4.9) and (4.10) hold. \square

We can now state and prove the Noether Theorem in its two parts, direct and inverse.

Theorem 4.7 (Noether Theorem – first part). *Let α be an m -form of Poincaré-Cartan type on $J^k(E)$. If X is a (weak) infinitesimal symmetry for $\mathcal{I}_{[\alpha]}$ on an open set $\mathcal{U} \subset J^k(E)$ with $\mathcal{L}_X \alpha$ holonomic, then the variational class of the $(m-1)$ -form $\eta^{(X)} := \iota_X \alpha$ satisfies a conservation law for the action $\mathcal{I}_{[\alpha]}$.*

Proof. Since α is of Poincaré-Cartan type, there is a holonomic $(m+1)$ -form λ such that $\beta = d\alpha + \lambda$ is a source form. Hence, if $\mathcal{L}_X \alpha$ is holonomic, then for any section $\sigma : \mathcal{V} \subset \mathbb{R}^m \rightarrow \mathcal{U} \subset E$ and any regular domain $D \subset \mathcal{V}$

$$\begin{aligned} \int_{\sigma^{(k)}(\partial D)} \eta^{(X)} &= \int_{\sigma^{(k)}(D)} d(\iota_X \alpha) = \\ &= \int_{\sigma^{(k)}(D)} \mathcal{L}_X \alpha - \int_{\sigma^{(k)}(D)} \iota_X d\alpha \stackrel{\mathcal{L}_X \alpha \text{ and } \lambda \text{ holon.}}{=} - \int_{\sigma^{(k)}(D)} \iota_X \beta. \end{aligned} \quad (4.11)$$

Since $\int_{\sigma^{(k)}(D)} \iota_X \beta = 0$ for any solution σ of the variational principle, $[\eta^{(X)}]$ satisfies a conservation law. \square

Now, before getting into the second part of Noether Theorem, we need to introduce an appropriate definition of regularity for Euler-Lagrange equations.

Let $[\alpha]$ be a variational class of m -forms on an open subset of $J^k(E)$ and assume that $\beta = \beta_i \omega_0^i \wedge dx^1 \wedge \dots \wedge dx^m$ is a source form on some open subset $\mathcal{W} \subset J^k(E)$. Assume also that β is of order $k_\beta \leq k-1$ (we may always reduce to this case by pulling back β on some jet space of higher order) and consider the differentials $d\beta_i$ of the components β_i of β . By assumptions, these differentials are equal to

$$d\beta_i = \sum_{j=1}^m \frac{\partial \beta_i}{\partial x^j} dx^j + \sum_{\substack{1 \leq j \leq n \\ 0 \leq |I| \leq k-1}} \frac{\partial \beta_i}{\partial y_I^j} dy_I^j = \sum_{j=1}^m \frac{d\beta_i}{dx^j} dx^j + \sum_{\substack{1 \leq j \leq n \\ 0 \leq |I| \leq k-1}} \frac{\partial \beta_i}{\partial y_I^j} \omega_I^j.$$

Due to this, for any section $\sigma : \mathcal{U} \rightarrow E$ whose k -th order lift $\sigma^{(k)}$ takes values in \mathcal{W} , we have

$$d \left(\beta_i(\sigma^{(k)}(x^1, \dots, x^m)) \right) = d\beta_i \left(\sigma_*^{(k)} \left(\frac{\partial}{\partial x^j} \right) \right) = \sum_{j=1}^m \frac{d\beta_i}{dx^j} \Big|_{\sigma^{(k)}(t)} dx^j.$$

Hence, σ is a solution of the Euler-Lagrange equations, i.e.

$$\beta_i(\sigma^{(k)}(x^1, \dots, x^m)) = 0, \quad 1 \leq i \leq n, \quad (4.12)$$

if and only if it is also a solution to the (expanded) system

$$\beta_i(\sigma^{(k)}(x^1, \dots, x^m)) = \frac{d\beta_i}{dx^j}(\sigma^{(k)}(x^1, \dots, x^m)) = 0, \quad 1 \leq j \leq m, 1 \leq i \leq n. \quad (4.13)$$

The system (4.13) is called *first prolongation* of (4.12). Note that if the 0-forms (= functions) β_i are of order k_β ($\leq k-1$), then, generically, the functions that define (4.13) are 0-forms of order $k_\beta + 1$.

Iterating this argument ($k' - k_\beta$) times for some $k' \leq k$, we get that (4.12) is equivalent to the expanded system

$$\begin{aligned} \beta_i(\sigma^{(k)}(x^\ell)) &= \frac{d\beta_i}{dx^{j_1}}(\sigma^{(k)}(x^\ell)) = \dots = \\ &= \left(\frac{d}{dx^{j_1}} \left(\frac{d}{dx^{j_2}} \dots \left(\frac{d}{dx^{j_{k'-k_\beta}}}(\beta_i) \right) \dots \right) \right) (\sigma^{(k)}(x^\ell)) = 0 \end{aligned} \quad (4.14)$$

for all $1 \leq j_h \leq m$ and $1 \leq i \leq n$. This new system is called *full prolongation* of (4.12) up to order k' . Note that, generically, the 0-forms that give the full prolongation up to order k' are 0-forms of order k' .

Definition 4.8. For a given system of Euler-Lagrange equations (4.12) of order k_β , let $F_\beta^{(k')}$ be the smooth map

$$F_\beta^{(k')} : \mathcal{W} \subset J^k(E) \rightarrow \mathbb{R}^N, \quad F_\beta^{(k')} := \left(\beta_i, \frac{d^{|I|}\beta_i}{dx^I} \right)_{\substack{1 \leq i \leq m \\ 1 \leq |I| \leq k' - k_\beta}}. \quad (4.15)$$

Here, $N = n \left(1 + \sum_{r=1}^{k'-k_\beta} \binom{m+r-1}{r} \right)$ and $\frac{d^{|I|}\beta_i}{dx^I}$, $I = (I_1, \dots, I_m)$, stands for

$$\frac{d^{|I|}\beta_i}{dx^I} := \underbrace{\frac{d}{dx^1} \frac{d}{dx^1} \dots \frac{d}{dx^1}}_{I_1\text{-times}} \underbrace{\frac{d}{dx^2} \frac{d}{dx^2} \dots \frac{d}{dx^2}}_{I_2\text{-times}} \dots \underbrace{\frac{d}{dx^m} \frac{d}{dx^m} \dots \frac{d}{dx^m}}_{I_m\text{-times}} \beta_i.$$

Let $Z_\beta^{(k')} := \{ u \in \mathcal{W} : F_\beta^{(k')}(u) = 0 \} \subset J^k(E)$. We say that the system of Euler-Lagrange equations (4.12) is *k' -regular on \mathcal{W}* if:

- i) the k' -th order jets of the solutions of the Euler-Lagrange equations constitute a dense subset of $\pi_{k'}^k(Z_\beta^{(k')}) \subset J^{k'}(E)$ and
- ii) the map $F_\beta^{(k')}$ is a submersion at all points of $Z_\beta^{(k')}$.

We are now able to state and prove the second part of Noether Theorem.

Theorem 4.9 (Noether Theorem – second part). *Let $k_o \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$ and α be an m -form of Poincaré-Cartan type on $J^k(E)$ of order less than or equal to $k_o - 1$. Assume that there exists an open subset $\mathcal{W} \subset J^k(E)$ admitting a system of adapted coordinates $\widehat{\xi}^{(k)} = (x^j, y^i, y_I^i)$, where the following non-degeneracy conditions are satisfied:*

a) the source form in $[d\alpha|_{\mathcal{W}}]$

$$\beta = \beta_i dy^i \wedge dx^1 \wedge \dots \wedge dx^m = \beta_i \omega_0^i \wedge dx^1 \wedge \dots \wedge dx^m$$

is of order $k_\beta \leq k_o$ and the system of Euler-Lagrange equations (4.12) is k_o -regular on \mathcal{W} ;

b) $\alpha\left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m}\right)\big|_u \neq 0$ at all u 's in \mathcal{W} .

Then, if η is an $(m-1)$ -form on \mathcal{W} of order less than or equal to $k_o - 1$, such that $[\eta]$ satisfies a conservation law for $\mathcal{I}_{[\alpha]}$, there exists a neighbourhood \mathcal{U} of $Z_\beta^{(k_o)} = \{u \in \mathcal{W} : F_\beta^{(k_o)}(u) = 0\}$ on which there are

- 1) a weak \mathcal{I} -symmetry X for $\mathcal{I}_{[\alpha]}$ with $\mathcal{L}_X \alpha$ holonomic and
- 2) an $(m-1)$ -form \mathfrak{z} that vanishes identically on any $(m-1)$ -tuple of vectors in a tangent space of a holonomic submanifold $\sigma^{(k)}(\mathcal{V})$ of a solution σ to the variational principle,

such that

$$\eta|_{\mathcal{U}} = \iota_X \alpha + \mathfrak{z} . \quad (4.16)$$

Proof. By Propositions 4.4 and 4.6, it suffices to prove the existence of a smooth \mathbb{R}^{m+n} -valued map $\mathbf{v} = (\mathbf{v}_B^j, \mathbf{v}^i) : \mathcal{U} \rightarrow \mathbb{R}^{m+n}$ on a neighbourhood \mathcal{U} of $Z_\beta^{(k_o)}$ and of an $(m-1)$ -form \mathfrak{z} on \mathcal{U} , such that: i) $\mathfrak{z}(Y_1, \dots, Y_m)|_{\sigma^{(k)}(\mathcal{V})} = 0$ for any choice of vector fields Y_i tangent to submanifolds $\sigma^{(k)}(\mathcal{V})$ of solutions σ of the Euler-Lagrange equations; ii) the following equations are satisfied

$$\iota_{X_{\mathbf{v}}} \alpha = \eta - \mathfrak{z} , \quad (4.17)$$

$$\beta\left(X_{\mathbf{v}}, \frac{d}{dx^1}, \dots, \frac{d}{dx^m}\right) = -d\eta\left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m}\right) + d\mathfrak{z}\left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m}\right) , \quad (4.18)$$

$$\begin{aligned} \mathcal{L}_{X_{\mathbf{v}}} \alpha & \left(\frac{d}{dx^{j_1}}, \dots, \frac{d}{dx^{j_r}}, \frac{\partial}{\partial y_{J_1}^{j_1}}, \dots, \frac{\partial}{\partial y_{J_{m-r}}^{j_{m-r}}} \right) = \\ & = d(\iota_{X_{\mathbf{v}}} \alpha) \left(\frac{d}{dx^{j_1}}, \dots, \frac{d}{dx^{j_r}}, \frac{\partial}{\partial y_{J_1}^{j_1}}, \dots, \frac{\partial}{\partial y_{J_{m-r}}^{j_{m-r}}} \right) = 0 \end{aligned} \quad (4.19)$$

for all $1 \leq r \leq m-1$, $1 \leq i_h \leq m$, $1 \leq j_\ell \leq r$ and $|J_\ell| = k$. Let us write α , η , \mathfrak{z} and β as sums of homogeneous forms, that is as

$$\begin{aligned} \alpha & = \alpha_0 dx^1 \wedge \dots \wedge dx^m + \\ & + \sum_{\substack{0 \leq |I| \leq k-1 \\ 1 \leq i \leq n, 1 \leq j \leq m}} \alpha_{i|j}^I \omega_I^i \wedge dx^1 \wedge \dots \widehat{}_j \dots \wedge dx^m + \lambda^{(\alpha)} \end{aligned} \quad (4.20)$$

$$\eta = \sum_{1 \leq j \leq m} \eta_j dx^1 \wedge \dots \widehat{j} \dots \wedge dx^m + \mu^{(\eta)} \quad (4.21)$$

$$\mathfrak{z} = \sum_{1 \leq j \leq m} \mathfrak{z}_j dx^1 \wedge \dots \widehat{j} \dots \wedge dx^m + \mu^{(\mathfrak{z})} \quad (4.22)$$

$$\beta = \sum_{1 \leq i \leq n} \beta_i \omega_0^i \wedge dx^1 \wedge \dots \wedge dx^m, \quad (4.23)$$

where $\lambda^{(\alpha)}, \mu^{(\eta)}, \mu^{(\mathfrak{z})}$ are holonomic forms, given by the terms in α, η and \mathfrak{z} , respectively, that are homogeneous of bidegree (ℓ, h) with $\ell \leq m - 2$. We now observe that, for any vector field X , the $(m - 1)$ -form $\iota_X \lambda^{(\alpha)}$ is holonomic. Hence, the $(m - 1)$ -form \mathfrak{z} has the prescribed properties if and only if $\mathfrak{z}' := \mathfrak{z} + \iota_X \lambda^{(\alpha)} - \mu^{(\eta)}$ has those properties. This yields that the above conditions are satisfied by \mathfrak{z}, α and η if and only if they are satisfied by $\mathfrak{z}', \alpha' = \alpha - \lambda^{(\alpha)}$ and $\eta' = \eta - \mu^{(\eta)}$. Due to this, *we may safely assume $\lambda^{(\alpha)} = 0$ and $\mu^{(\eta)} = 0$.*

Consider now the function

$$g : \mathcal{W} \longrightarrow \mathbb{R}, \quad g(u) := d\eta \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) \Big|_u.$$

We claim that g vanishes identically on $Z_\beta^{(k_o)}$. Indeed, by assumption (a), the k_o -th order jets of solutions σ to the Euler-Lagrange equations form a dense subset \tilde{Z} of $\pi_{k_o}^k(Z_\beta^{(k_o)})$. In particular, for any $\tilde{u} \in \tilde{Z}$ and $1 \leq i \leq m$, we have that

$$\frac{d}{dx^i} \Big|_{\tilde{u}} - \sigma_*^{(k_o)} \left(\frac{\partial}{\partial x^i} \Big|_p \right) \in \text{Span} \left\{ \frac{\partial}{\partial y_J^j} \Big|_u, |J| = k_o \right\} \subset T_{\tilde{u}} J^{k_o}(E),$$

for some solution σ with $\sigma^{(k_o)}(p) = \tilde{u}$. Since $[\eta]$ satisfies a conservation law and η is of order less than or equal to $k_o - 1 < k_o$, we get that for any jet $u \in (\pi_{k_o}^k)^{-1}(\tilde{Z}) \subset Z_\beta^{(k_o)}$

$$g(u) := d\eta \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) \Big|_u = d\eta \left(\sigma_*^{(k)} \left(\frac{\partial}{\partial x^1} \Big|_p \right), \dots, \sigma_*^{(k)} \left(\frac{\partial}{\partial x^m} \Big|_p \right) \right) = 0.$$

By continuity of g , we get $g(u) = 0$ for any $u \in Z_\beta^{(k_o)}$.

Since we are also assuming that $F_\beta^{(k_o)} : \mathcal{W} \longrightarrow \mathbb{R}^N$ is a submersion at any $u \in Z_\beta^{(k_o)}$ and that η is of order less than or equal to $k_o - 1$, by standard properties of submanifolds (see e.g., [14], Lemma 2.1 and [16], Prop. 2.10), there exist an open neighbourhood $\mathcal{U} \subset \mathcal{W}$ of $Z_\beta^{(k_o)}$ and some (non-uniquely determined) smooth functions $\widehat{\mathbf{v}}_I^j$ on \mathcal{U} , with $1 \leq j \leq n$, $0 \leq |I| \leq k_o - k_\beta$,

such that

$$g = d\eta \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) = \sum_{\substack{0 \leq |I| \leq k_o - k_\beta \\ 1 \leq j \leq n}} \widehat{\mathbf{v}}_I^j \frac{d^{|I|} \beta_j}{dx^I} . \quad (4.24)$$

Further, since the left-hand side of (4.24) is a function of the coordinates (x^i, y_I^j) with $|I| \leq k_o$, there is no loss of generality if we assume that all functions $\widehat{\mathbf{v}}_I^j$ depend only on the coordinates (x^i, y_I^j) with $|I| \leq k_o$.

Now, if $k_o > k_\beta$, let $\mathfrak{z}^{(1)}$ be the $(m-1)$ -form on \mathcal{U} defined by

$$\mathfrak{z}^{(1)} := \sum_{\substack{|I|=k_o-k_\beta \\ 1 \leq j \leq n, 1 \leq i \leq m}} (-1)^{i-1} \widehat{\mathbf{v}}_I^j \frac{d^{(k_o-k_\beta-1)} \beta_j}{dx^{I-1_i}} dx^1 \wedge \dots \widehat{}_i \dots \wedge dx^m . \quad (4.25)$$

By construction, the integrals of the $(m-1)$ -form and of its exterior differential $d\mathfrak{z}^{(1)}$ along the holonomic submanifolds $\sigma^{(k)}(\mathcal{V})$ of solutions $\sigma : \mathcal{V} \rightarrow E$ of the variational principle are identically equal to 0. Indeed, this is a consequence of the fact that, modulo holonomic terms, the components of $\mathfrak{z}^{(1)}$ and of $d\mathfrak{z}^{(1)}$ are given by linear combinations of the components of the map $F_\beta^{(k_o)}$. Furthermore, we have

$$\sum_{\substack{|I|=k_o-k_\beta \\ 1 \leq j \leq n}} \widehat{\mathbf{v}}_I^j \frac{d^{k_o-k} \beta_j}{dx^I} = d\mathfrak{z}^{(1)} \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) - g^{(1)} \quad \text{with}$$

$$g^{(1)} := \sum_{\substack{|I|=k_o-k_\beta \\ 1 \leq j \leq n, 1 \leq i \leq m}} \frac{d\widehat{\mathbf{v}}_I^j}{dx^i} \frac{d^{k_o-k_\beta-1} \beta_j}{dx^{I-1_i}} + \sum_{1 \leq |J| \leq k_o-k_\beta-1} \widehat{\mathbf{v}}_I^j c_{iI-1_i}^J \frac{d^{|J|} \beta_j}{dx^J} ,$$

where we used the notation $c_{iI-1_i}^J$ to indicate the unique functions that satisfy the identity

$$\frac{d^{|I|}}{dx^I} = \frac{d}{dx^i} \frac{d^{|I|-1}}{dx^{I-1_i}} + \sum_{0 \leq |J| \leq |I|-1} c_{iI-1_i}^J \frac{d^{|J|}}{dx^J} .$$

Combining this with (4.24), we get the existence of functions $\widehat{\mathbf{v}}_I'^j$ on \mathcal{U} , which depend only on the coordinates (x^i, y_I^j) with $|I| \leq k_o + 1$ and such that

$$d(\eta - \mathfrak{z}^{(1)}) \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) = \sum_{\substack{0 \leq |I| \leq k_o-k_\beta-1 \\ 1 \leq j \leq n}} \widehat{\mathbf{v}}_I'^j \frac{d^{|I|} \beta_j}{dx^I} . \quad (4.26)$$

Iterating this construction, we obtain a sequence of $(m-1)$ -forms $\mathfrak{z}^{(1)}, \mathfrak{z}^{(2)}, \dots, \mathfrak{z}^{(k_o-k_\beta)}$ such that

$$d(\eta - \mathfrak{z}^{(1)} - \dots - \mathfrak{z}^{(k_o-k_\beta)}) \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) = \sum_{i=1}^n \widehat{\mathbf{v}}^i \beta_i , \quad (4.27)$$

for some appropriate smooth functions $\widehat{\mathbf{v}}^i : \mathcal{U} \rightarrow \mathbb{R}$ on an open neighbourhood \mathcal{U} of $Z_\beta^{(k_o)}$, which depend only on the coordinates (x^i, y_I^j) with $|I| \leq 2k_o - k_\beta \leq 2k_o$. Note that, by construction of such $(m-1)$ -forms $\mathfrak{z}^{(i)}$, their integrals along holonomic submanifolds $\sigma^{(k)}(\mathcal{V})$ of solutions $\sigma : \mathcal{V} \rightarrow E$ to the variational principle, are identically equal to 0.

Set $\mathfrak{z} = \mathfrak{z}^{(1)} + \dots + \mathfrak{z}^{(k_o - k_\beta)}$ and, for any $i = 1, \dots, m$, denote by h_i the function

$$h_i := \eta_i - \mathfrak{z}_i - \sum_{\substack{0 \leq J \leq |k - k_o - 1| \\ 1 \leq j \leq n}} \mathfrak{z} \left(\frac{\partial}{\partial y_I^j}, \frac{d}{dx^1}, \dots, \widehat{\frac{d}{dx^i}} \dots \frac{d}{dx^m} \right)$$

where we denote by η_i and \mathfrak{z}_i the coefficients of the $(m-1)$ -forms $dx^1 \wedge \dots \widehat{dx^i} \dots \wedge dx^m$ in the expansions of η and \mathfrak{z} . Then, consider the $(m+n)$ -tuple of functions $\mathbf{v} = (\mathbf{v}_B^i, \mathbf{v}^j)$ defined by

$$\mathbf{v}_B^i := \frac{1}{\alpha_0} \left((-1)^{i+1} \sum_{\substack{0 \leq J \leq |k - k_o - 1| \\ 1 \leq j \leq n}} \alpha_{i|j}^J \left(\frac{d}{dx^1} \right)^{J_1} \dots \left(\frac{d}{dx^m} \right)^{J_m} (\widehat{\mathbf{v}}^i) + (-1)^i h_i \right),$$

$$\mathbf{v}^i := y_{1_r}^i \mathbf{v}_B^r - \widehat{\mathbf{v}}^i.$$

One can directly check that the $(m+n)$ -tuple $\mathbf{v} = (\mathbf{v}_B^i, \mathbf{v}^j)$ satisfies the equation

$$(-1)^i \alpha_0 \mathbf{v}_B^i + \sum_{\substack{0 \leq J \leq |k - k_o - 1| \\ 1 \leq j \leq n}} \alpha_{i|j}^J \left(\frac{d}{dx^1} \right)^{J_1} \dots \left(\frac{d}{dx^m} \right)^{J_m} (\mathbf{v}^i - y_{1_r}^i \mathbf{v}_B^r) = h_i. \quad (4.28)$$

From this, the expressions (4.20) - (4.23) together with the assumptions $\lambda^{(\alpha)} = \mu^{(\eta)} = 0$ and the identity (4.27), one gets that $X_{\mathbf{v}}$ satisfies

$$\iota_{X_{\mathbf{v}}} \alpha = \eta|_{\mathcal{U}} - \mathfrak{z}, \quad (4.29)$$

$$\begin{aligned} \iota_{X_{\mathbf{v}}} \beta \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right) &= (\mathbf{v}^i - y_{1_r}^i \mathbf{v}_B^r) \beta_i = - \sum_{i=1}^n \widehat{\mathbf{v}}^i \beta_i = \\ &= d(-\eta|_{\mathcal{U}} + \mathfrak{z}) \left(\frac{d}{dx^1}, \dots, \frac{d}{dx^m} \right). \end{aligned} \quad (4.30)$$

Hence (4.17) and (4.18) hold. It remains to show that also equations (4.19) are satisfied. To show this, we observe that from (4.29) and the construction of the $(m-1)$ -form \mathfrak{z} , the $(m-1)$ -form $\iota_X \alpha$ is an m -form of order less than or equal to $2k_o - k_\beta$. Hence $d(\iota_X \alpha)$ is an m -form of order less than or equal to $2k_o - k_\beta + 1 \leq 2k_o + 1$. Since $k_o \leq \left\lfloor \frac{k}{2} \right\rfloor - 1$, it follows that $d(\iota_X \alpha)$ is of order less than or equal to $k - 1$ and that the contraction of $d(\iota_X \alpha)$ with any vector field $\frac{\partial}{\partial y_I^j}$ with $|J| = k$ is 0. From this, (4.19) follows. \square

5. AN EXAMPLE OF A FORM OF POINCARÉ-CARTAN TYPE

Let $\mathcal{E} = \mathbb{R}^{1,3}$ be the space-time of Special Relativity and denote by (x^0, x^1, x^2, x^3) and

$$\eta := \eta_{ij} dx^i \otimes dx^j, \quad \text{with } (\eta_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

the standard coordinates and the standard flat metric of $\mathbb{R}^{1,3}$, respectively (as usual, we follow the classical Einstein convention on summations). In Special Relativity the electromagnetic field is represented by a closed 2-form $\mathbb{F} = F_{ij} dx^i \wedge dx^j$, that is a 2-form which can be locally written as

$$\mathbb{F} = d\mathbb{A} = \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right) dx^i \wedge dx^j, \quad (5.1)$$

for a 1-form $\mathbb{A} = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$, called *4-potential*. Since $\mathcal{E} = \mathbb{R}^{1,3}$ is contractible, we may assume that \mathbb{A} is globally defined and consider the Maxwell equations in the vacuum as partial differential equations of second order on the 4-potential \mathbb{A} . It is well known that these equations are precisely the Euler-Lagrange equations for the Lagrangian

$$L : J^1(T^*\mathcal{E}) \longrightarrow \mathbb{R}, \quad L(j^1(\mathbb{A})) := -\frac{1}{16\pi c} |\mathbb{F}|_\eta^2 = -\frac{1}{16\pi c} \eta^{i\ell} \eta^{jm} F_{ij} F_{\ell m}, \quad (5.2)$$

where F_{ij} are the coordinate components of $\mathbb{F} = d\mathbb{A}$, $(\eta^{ij}) = (\eta_{\ell m})^{-1}$ and c is the physical constant given by the speed of light.

Let $E := T^*\mathcal{E}$ and denote by $\widehat{\xi}^{(2)} = (x^i, A_m, A_{m,n}, A_{m,nr})$ a (global) system of adapted coordinates on the second order jet space $J^2(E) = J^2(T^*\mathcal{E})$. If we denote by α_L the 4-form on $J^2(E)$

$$\alpha_L = L(x^i, A_m, A_{m,n}) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

with L defined in (5.2), we have that a 1-form $\mathbb{A} : \mathcal{E} \simeq \mathbb{R}^4 \longrightarrow T^*\mathcal{E}$ satisfies Maxwell's equations if and only if it satisfies the variational principle of the action $\mathcal{I}_{[\alpha_L]}$.

The proof of Prop. A2 in [19] (see also [4], Thm.1.3.11) gives an algorithm to determine m -forms of Poincaré-Cartan type in a given variational class. For the variational class $[\alpha_L]$, such algorithm produces the 4-form

$$\begin{aligned} \alpha &= L dx^0 \wedge \dots \wedge dx^3 + \sum_{h=0}^3 (-1)^h \frac{\partial L}{\partial A_{k,h}} dx^0 \wedge \dots \wedge \widehat{} \dots \wedge dx^3 \wedge (dA_k - A_{k,r} dx^r) = \\ &= \left(\frac{1}{16\pi c} F^{kh} F_{kh} \right) dx^0 \wedge \dots \wedge dx^3 - \sum_{h=0}^3 (-1)^h \frac{1}{4\pi c} F^{kh} dx^0 \wedge \dots \wedge \widehat{} \dots \wedge dx^3 \wedge dA_k. \end{aligned} \quad (5.3)$$

One can directly check that the differential $d\alpha$ is a source form (modulo a holonomic 5-form) whose components are precisely the terms of the Euler-Lagrange equations of L .

As is well known, Maxwell equations are conformally invariant. This corresponds to the fact that, for each $k \geq 1$, and for each conformal Killing vector field X of $\mathbb{R}^{1,3}$, the corresponding vector field $\widehat{X}^{(k)}$ on $J^k(T^*\mathcal{E})$, whose local flows $\Phi_t^{\widehat{X}^{(k)}} \in \text{Diff}_{\text{loc}}(J^k(T^*\mathcal{E}))$ are the natural lifts of the local flows $\Phi_t^X \in \text{Diff}_{\text{loc}}(\mathcal{E})$ of X , is an infinitesimal symmetry for the action $\mathcal{I}_{[\alpha]}$. For instance, each vector field $\xi_{(j)} := \frac{\partial}{\partial x^j}$, $0 \leq j \leq 3$, generating the translations in the directions of the x^j -axis, is clearly a conformal Killing vector field and its associated vector field $\widehat{\xi}_{(j)}^{(1)} := \frac{\partial}{\partial x^j} + A_{j,k} \frac{\partial}{\partial A_k}$ on $J^1(T^*\mathcal{E})$ is an infinitesimal symmetry for $\mathcal{I}_{[\alpha]}$. One can directly check that (see e.g. [4])

$$\begin{aligned} \iota_{\widehat{\xi}_{(j)}^{(1)}} \alpha &= \iota_{V_{(j)}} dx^0 \wedge \dots \wedge dx^3 \quad \text{mod. holonomic 3-forms,} \\ \text{with } V_{(j)} &= \frac{1}{4\pi c} \left(F_{j\ell} F_{km} \eta^{\ell m} - \frac{1}{4} \eta^{r\ell} \eta^{sm} F_{rs} F_{\ell m} \right). \end{aligned} \quad (5.4)$$

Further, the components of $\iota_{V_{(j)}} dx^0 \wedge \dots \wedge dx^3$ coincide (up to sign) with the components T_{jr} , $0 \leq r \leq 3$, of the electromagnetic stress-energy tensor T . Hence, from this and Theorem 4.7, one has an alternative derivation of the following well-known property: the (equivalence classes of the) translational symmetries $\widehat{\xi}_{(j)}^{(1)}$, $0 \leq j \leq 3$, correspond via the Noether Theorem to the (equivalence classes of) the conserved stress-energy currents $\Phi_{(j)} := T_{js}$.

We now recall that in [1], Anco and Pohjanpelto classified all local conservation laws of Maxwell equations. There the authors proved that, modulo equivalences, any local conservation law is a linear combination of some special currents, constructed using conformal Killing vector fields and conformal Killing-Yano tensor fields. Using our proof of Theorem 4.9, one can determine the infinitesimal symmetries, which correspond to all such conservation laws through a contraction with the form α of Poincaré-Cartan type. From previous observations, it is reasonable to expect that such infinitesimal symmetries (and, consequently, most geometric properties of the 3-form (5.3)) are strongly related with the conformal Killing vector fields and, more interesting, with the conformal Killing-Yano tensor fields of $\mathbb{R}^{1,3}$. Making these relations explicit would very likely pave the way towards generalisations of various kind, quite useful for studying for instance Maxwell equations in curved spaces.

APPENDIX A. ERRATUM TO “LIE ALGEBRAS OF CONSERVATION LAWS OF VARIATIONAL ORDINARY DIFFERENTIAL EQUATIONS”

The purpose of this short appendix is to remove an incorrect claim of [6]. There, in Prop. 3.5, it is improperly stated that, when $\dim M \geq 2$,

the \mathcal{D} -symmetries coincide with the vector fields of the form $X = X_{\mathbf{v}}$ with $\partial \mathbf{v} / \partial y_{(k)}^i = 0$. The correct claim is that *the former are only a subset of the latter*.

This does not effect the results of the paper, provided that one considers *weak \mathcal{D} -symmetries* (see Definition 4.2 above) in place of \mathcal{D} -symmetries. We remark that weak \mathcal{D} -symmetries can be considered as truncations up to order k of \mathcal{D} -symmetries of $J^\infty(E)$ and that there is no difference between the two notions if one works on $J^\infty(E)$ in place of $J^k(E)$.

The correction imposes a few other minor adjustments, which one can immediately determine by looking at the more general results of the present paper. For instance, the hypothesis of Thm. 3.10 in [6] on the orders of conservation laws and the 1-form of Poincaré-Cartan type should be modified according to Theorem 4.9 of this paper, which includes and extends the previous.

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